Lowest Unique Bid Auctions with Signals

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Abstract

A lowest unique bid auction allocates a good to the agent who submits the lowest bid that is not matched by any other bid. This peculiar mechanism recently experienced a surge in popularity on the Internet. We study lowest unique bid auctions from a theoretical point of view and show that in equilibrium the mechanism is unprofitable for the seller unless there exist sizeable gains from trade. But we also show that it can become highly lucrative if some bidders are myopic. In this second case, we analyze the key role played by a number of private signals bidders receive about the current status of the bids they submitted.

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1 Introduction

A new wave of websites has been intriguing consumers on the Internet over the very last few years. These websites sell goods of considerable value (electronic equipment, watches, holidays, and even cars and houses) through quite a peculiar auction mechanism: the winner is the bidder who submits the lowest *unique* offer, i.e., the lowest offer among the ones that are not matched by any other bid. Such a mechanism is commonly called a lowest unique bid auction (LUBA) and leads to impressively low selling prices. Just to mention some examples, an iPod (valued at 200 Euros) has been sold for 0.25 Euros, a Sony Playstation 3 (400 €) for 0.81 €, and a new Volkswagen Beetle Cabriolet (32,000 €) for 32.83 €. These are not exceptions. As a rule of thumb, goods are usually sold for a price that is 0.1-0.3% of the market value. It is not through the selling price that the auctioneer thus expects to make profits. These rather stem from the accrual of participants' bidding fees as bidders have to pay a fixed cost for any bid they submit.[1]

Websites offering LUBAs first appeared in Scandinavian countries in 2007. Since then, they progressively spread in many other countries following what seems to be quite a common pattern. Initially, early entrants benefit from consumers' enthusiasm and participation and are thus able to enjoy substantial profits. Then, a sharp decline in profitability occurs as competition gets tougher, consumers rapidly lose their interest as they learn that not all that glitters is gold, and legal controversies about the nature of the business often arise. Operators thus rapidly leave the market which basically closes down. LUBAs websites already went through the entire sequence in most European countries. On the contrary, there currently are active websites offering LUBAs in many other countries including the US, the UK, India and Australia.

But a part from the study of the profitability and sustainability of LUBAs, there are other reasons that justify the analysis of this auction format. The selling mechanism is in fact very interesting from a game-theoretic point of view as bidders face an unusual and rather complex strategic situation. Moreover, the theoretical predictions stemming from

[1] We stress that websites that employ lowest unique bid auctions are not platforms that aim to match buyers and sellers (such as for instance eBay). On the contrary, in the case of LUBAs, the owner of the website and the (unique) seller/auctioneer coincide.

such an analysis can be compared with the actual bidding behavior that can be observed "in the field". Finally, optimal design remains an issue as LUBAs can turn out to be useful in other contexts such as marketing campaigns as well as charities and fund-raising activities.[2]

This paper investigates some theoretical properties of LUBAs. We focus in particular on the characterization of agents' equilibrium behavior, on the study of the profitability of the mechanism, and on the analysis of the role played by a number of signals that the auctioneer sends to the bidders during the course of the auction. We analyze LUBAs in two different scenarios: in the first one, all the players are assumed to be fully rational; the second scenario allows instead for the presence of some boundedly rational bidders.

But before tackling these issues, let us introduce in more detail the functioning of a LUBA. As a first step, customers must register to one of the websites that employ this selling mechanism and transfer some money to a personal deposit. Users can then browse through the items on sale and submit as many bids as they want. Bids are expressed in cents and are private. Every time that a user places a bid, a fixed amount of money (around 1 Euro/Dollar) is deducted from his deposit. The auctioneer justifies this cost as the price for a "packet of information" that he sends to the bidder. In fact, as soon as a bid is submitted, the user receives one of the three following messages (or signals): 1) Your bid is currently the unique lowest bid; 2) Your bid is unique but is not the lowest; or 3) Your bid is not unique. During the bidding period, which usually lasts for a few days, users can at any time log in to their account to check the current status of their bids (signals are always updated), place new bids, or refill their deposit. Once the auction closes, the object is sold to the bidder who submitted the lowest unique bid. For instance, if agents a and b offer 1 cent, c offers 2 cents, a and d offer 3 cents, and e offers 6 cents, then the object is sold to c for a price of 2 cents.

This allocation mechanism is, therefore, considerably different in respect to traditional auction formats.[3] Yet, there are still few very recent papers that explicitly study various

[2] The study of how different auction mechanisms can be used in charities and fundraising activities recently became an active area of research (see for instance Engers and Mcmanus, 2007 and Schram and Onderstal, 2009).

[3] See Klemperer (1999) or Krishna (2002) for detailed reviews of standard auction theory. On the other hand, LUBAs belong to the family of so-called "pay-per-bid auctions". Other examples include "penny

versions of unique bid auctions.[4] Houba *et al.* (2011) and Rapoport *et al.* (2009) analyze the equilibria of a LUBA in which bidders submit a unique bid, there is a non-negative bidding fee, and the winner pays his bid. Both papers find that in the symmetric mixed equilibrium, bidders randomize with decreasing probabilities over a support that comprises the lowest possible bid and is made of consecutive numbers.[5] Östling *et al.* (2011) obtained a similar result for what they call a LUPI (Lowest Unique Positive Integer) game in which players can again submit a single bid, but there are no bidding fees, and the winner does not have to pay his bid. The peculiarity of this study is that the number of participants is assumed to follow a Poisson distribution. Finally, Eichberger and Vinogradov (2008) analyzed a LUBA (which they called a LUPA, i.e., Least Unmatched Price Auction) where bidders can submit multiple costly bids, and the winner must pay his winning bid. Given that no information about other bidders' behavior is released during the auction, they model the game as a simultaneous one. For some special ranges of the parameters, they show the existence of a unique Nash equilibrium in which agents mix over bidding strings that comprise the minimum allowed bid and are made of consecutive numbers. In addition to the theoretical analysis, the papers by Houba *et al.* (2011) and by Rapoport *et al.* (2009) propose some algorithms for computing the symmetric mixed strategy equilibrium. Instead, the papers by Östling *et al.* (2011) and Eichberger and Vinogradov (2008) have an empirical part, which is based on field and/or experimental data. Theoretical predictions find some empirical evidence at the aggregate level but a lower one at the individual level.

With respect to this ongoing literature, our paper differs in some important ways. The main novelty is in the analysis of the role played by the signals that the bidders receive about the status of their submitted bids. So far, this aspect has been neglected. We study

auctions" and "price reveal auctions". In a penny auction (see for instance Augenblick, 2012 or Hinnosaar, 2010) each bid increases the current price by a fixed amount (a penny) and restarts a public countdown; the winner is the bidder who holds the winning bid at the moment the countdown expires. A price reveal auction (see for instance Gallice, 2012) is instead a descending price auction in which bidders can observe the current price of the item only by paying a fee. Every time a player observes the price this falls by a predetermined amount and the auction ends as soon as a bidder buys the item.

[4] Indeed, none of these papers (and ours will be no exception) *exactly* mimics all the features that characterize a LUBA. A common trait of these studies is, in fact, the necessity to introduce some simplifying assumptions as the analysis of a fully fledged model is too complex.

[5] Rapoport *et al.* (2009) also analyzed HUBAs, i.e., unique bid auctions in which the winner is the bidder who submits the highest unmatched offer. Such a mechanism was first studied by Raviv and Virag (2009).

how these signals influence agents' bidding strategies, and we show them to be a key element of the mechanism, especially for what concerns out of equilibrium play. Second, we explicitly model bidders' decisions about how much to invest in the auction (i.e., how many bids to submit). We frame the problem as a probabilistic contest, and we study how the optimal level of investment depends on bidders' subjective and private valuations (another peculiarity of the paper as all other studies assume a common valuation) as well as on other parameters of the game. Third, by modeling LUBAs as a sequential game that captures the actions of both the bidders and the auctioneer, we solve for the seller's optimal choices about the amount of the bidding fee and the duration of the auction.

Finally, we investigate the expected profitability of the mechanism. We show that if agents are rational then the expected profits of the auctioneer can be positive only if his valuation of the good is (much) lower than the expected valuation of the bidders. In other words, the mechanism is profitable only if there are large gains from trade. This would imply that websites offering LUBAs should not have proliferated the way they do. To rationalize this observation, we show how a LUBA can become highly profitable when some of the bidders lack the necessary commitment to stick to equilibrium strategies and thus get easily trapped in a costly war of attrition.

The remainder of the paper is organized as follows: Section 2 formalizes the strategic situation and characterizes its equilibrium under the assumption of perfect rationality of the players. Section 3 investigates what may happen when some of the bidders are boundedly rational. Section 4 concludes. The proofs of all the propositions are collected in the appendix.

2 The model

We introduce and analyze a sequential game which displays the key aspects of a lowest unique bid auction. The game features $(n+1)$ risk-neutral players: an auctioneer (a) who must decide if to auction a certain good through a LUBA and a set $N = \{1, ..., n\}$ of potential buyers. To capture the auctioneer's superior information about the level of participation (as it has already been mentioned, agents who want to participate in a LUBA must first register to the auctioneer's website), we assume that the auctioneer knows n with certainty while potential buyers only know that $n = n_o + 1$ where n_o (i.e., the number of other participants) is a random variable distributed over $\{0, ..., n_{\max}\}$ according to the probability distribution g.

Let $v_a \geq 0$ be the value of the good for the auctioneer and v_i be the subjective valuation of agent $i \in N$. We assume that agents' valuations are identically and independently distributed on the support $[0, v_{\max}]$, where $v_{\max} \geq v_r$ and v_r is the retail price of the item.[6] The realization v_i is private information of bidder i, but we let the mean of the distribution $\bar{v}$ be common knowledge.[7] We also assume that $v_a \leq v_r$ such that there could be gains from trade.[8]

The game spans over $T + 3$ periods with $t \in \{-1, 0, 1, ..., T, T + 1\}$. At period $t = -1$ the auctioneer decides if to open a LUBA. If this is the case, the auctioneer chooses and publicly announces T (which thus becomes common knowledge) and credibly commits to sell the good to the buyer that in period $t = T + 1$ (i.e., as soon as the LUBA has closed) holds the lowest unmatched positive bid. The selling price is given by that bid. Potential buyers must thus solve two distinct and subsequent problems. In the first, which takes place at $t = 0$ and which we label the "investment decision", each agent decides the maximum amount that he is willing to invest in the game. Given that the submission of each bid entails the payment of the fee $c > 0$, the investment decision determines the number of bids that an agent is willing to submit throughout the game.[9]

In the second problem, which we label the "bidding phase", each bidder must decide when and where to place these bids. The bidding phase starts at $t = 1$ (the opening of

[6] The retail price v_r is publicly known. But as usual this in general differs from an agent's personal valuation (or willingness to pay) v_i. Given that, as it has been mentioned in the introduction, the selling price in a LUBA is usually a tiny fraction of v_r, the mechanism in principle appeals not only to agents for which $v_i \leq v_r$ (i.e., people that would not buy the good at the market price because they find it too expensive) but also to those for which $v_i > v_r$ (i.e., agents that would enjoy a surplus even if they buy the good at price v_r but that are attracted by the possibility of buying the good at a much lower price).

[7] Notice that this is a weaker assumption with respect to the independent private value assumption that is commonly used in auction theory. In fact, the latter requires players to know the entire distribution of valuations and not just the mean value.

[8] The assumption captures the fact that the auctioneer may acquire from the producer the good that he will auction for less than its retail price because of quantity discounts and/or marketing reasons.

[9] We initially consider the specific amount of the bidding fee c to be exogenously given. In Section 2.3 we will relax this assumption and solve for the optimal fee c^* that the auctioneer should strategically choose in order to maximize his expected profits.

the auction) and ends at $t = T$ (the closing of the auction). In any period each player can place at most one bid. More precisely, at any $t \in \{1, ..., T\}$ each agent $i \in N$ plays $x_i^t \in \{\emptyset\} \cup \mathbb{N}^+$. Action $x_i^t = \emptyset$ indicates that agent i does not bid at period t. Action $x_i^t \neq \emptyset$ indicates that agent i submits at time t the bid $x_i^t \in \mathbb{N}^+$.[10]

At the end of each period, the auctioneer checks the current distribution of all the bids that the auction received up to that point. He then sends to each bidder a truthful and private signal about the current status of any bid that the agent has submitted up to that period. More precisely, if agent i plays $x_i^t \neq \emptyset$ then the agent receives a signal $\sigma^\tau\left(x_i^t\right) \in \{W, M, L\}$ in any period $\tau \in \{t, ..., T\}$. Notice that a bidder who did not submit any bid receives no signal while a bidder that submitted multiple bids receives multiple signals. It is common knowledge that the signals mean the following:

- $\sigma^\tau\left(x_i^t\right) = W$ indicates that x_i^t is currently the Winning bid (i.e., at time τ, x_i^t is the lowest unique bid).
- $\sigma^\tau\left(x_i^t\right) = M$ indicates that x_i^t Might become the winning bid (i.e., at time τ, x_i^t is a unique bid but it is not the lowest).
- $\sigma^\tau\left(x_i^t\right) = L$ indicates that x_i^t is a Losing bid (i.e., at time τ, x_i^t is not a unique bid).

The signals associated to a specific bid can thus change over time. In particular, the signal $\sigma^\tau\left(x_i^t\right) = W$ can be updated by $\sigma^r\left(x_i^t\right) = M$ (a bidder j places at time $r \in \{\tau + 1, ..., T\}$ a bid $x_j^r < x_i^t$ such that $\sigma^r(x_j^r) = W$) or by $\sigma^r\left(x_i^t\right) = L$ (a bidder j bids $x_j^r = x_i^t$). For similar reasons the signal $\sigma^\tau\left(x_i^t\right) = M$ can be updated by $\sigma^r\left(x_i^t\right) = W$ or by $\sigma^r\left(x_i^t\right) = L$. The signal $\sigma^\tau\left(x_i^t\right) = L$ cannot be updated because the status of a bid that is not unique cannot change as agents cannot move or cancel their bids once these have been submitted.

The LUBA employs a specific tie breaking rule to solve for the cases in which, conditional on entry having occurred, a unique bid does not exist. The rule specifies that if at time $t \in \{1, ..., T\}$ such an offer does not exist, then the current winner is the bidder who first submitted the lowest bid that was bid by the lowest number of agents. To cover all

[10]As it happens in actual LUBAs, the grid of admissible bids is discrete and it is defined in terms of Euro/Dollar cents. As such $x_i^t = 100$ indicates that in period t agent i places a bid of one Euro/Dollar.

possible cases, we add the (empirically irrelevant) rule that if two or more agents chose this bid simultaneously then the tie is broken randomly. As such, there is always a single bid that holds the signal W and thus there is always a unique provisional winner.

In order to formalize players' payoffs we let $\eta_i^t \in \mathbb{N}$ be the number of bids submitted by agent i up to period t included such that η_i^T is the number of bids submitted by i over the course of the entire auction. Finally, and abstracting from the identity of the bidder and the period in which it was submitted, we indicate with $\hat{x}$ the bid that wins the LUBA, i.e., the bid for which $\sigma^T(\hat{x}) = W$. Payoffs thus take the following form:

$$u_a^{T+1} = \begin{cases} \left(\sum_{i=1}^{n} \eta_i^T\right) c + \hat{x} - v_a & \text{if } a \text{ opens the auction} \\ 0 & \text{otherwise} \end{cases}$$

$$u_i^{T+1} = \begin{cases} v_i - \eta_i^T c - \hat{x} & \text{if } i \text{ wins the auction} \\ -\eta_i^T c & \text{otherwise} \end{cases} \quad \text{for } i \in N$$

Notice that the payoffs of the bidders comprise their outside option of not participating in the auction as $u_i^{T+1} = 0$ when $\eta_i^T = 0$. Similarly, we let the auctioneer's outside option of not opening the LUBA generate a null payoff.

In solving the game, we proceed backwards. We first define the equilibria of the bidding phase of the game (subsection 2.1). We then solve for the investment decision of the bidders (2.2). Finally, we analyze the equilibrium behavior of the auctioneer (2.3).

2.1 The bidding phase

Let $\eta_i^{\max} \in \mathbb{N}$ be the maximum number of bids that a rational bidder i is willing to submit in the LUBA. Section 2.2 will provide a rationale for such a formulation and explicitly derive $\eta_i^{\max}$ as a function of the agent's valuation v_i and of the parameters of the game. By now, we take $\eta_i^{\max}$ as given with $\eta_i^{\max} \geq 0$ for every i. A bidder with $\eta_i^{\max} = 0$ does not submit any bid. This means that he does not participate in the auction and realizes a final payoff $u_i^{T+1} = 0$. More interesting is the analysis of the bidding behavior of active players. These are the bidders with $\eta_i^{\max} \geq 1$, i.e., those that submit at least one bid. We indicate with $I \subseteq N$ the subset of active bidders. Every agent $i \in I$ must then decide when and where to place his bids.

As for the timing dimension, notice that the tie breaking rule implies that all the strategies such that bidders delay the submission of their bids are weakly dominated. We disregard such strategies and focus instead on the undominated strategies that prescribe players to submit their bids as soon as possible.[11] The following remark more precisely defines the set of players that submit a bid in generic period $t \in \{1, ..., T\}$.

Remark 1 *At any period $t \in \{1, ..., T\}$, a bidder $i \in I$ plays action $x_i^t \neq \emptyset$ (i.e., he submits a bid) if and only if $\eta_i^{t-1} < \eta_i^{\max}$ and $\sigma^{t-1}(x_i^r) \neq W$ for every element of the set $\{x_i^r\}_{r=1}^{t-1}$.*

The agents that submit an offer in period $t \in \{1, ..., T\}$ are thus those who did not run out of bids and do not hold the current winning offer. The bidder who owns the winning offer in period $t-1$ should in fact abstain from submitting additional bids given that these are costly and agents' final payoff is strictly decreasing in η_i^T. Only if the winning bid turns into a losing one (i.e., the signal $\sigma^{t-1}(x_i^r) = W$ is updated by $\sigma^{t-1+k}(x_i^r) \neq W$), the agent will restart submitting his remaining $\eta_i^{\max} - \eta_i^{t-1}$ bids.[12]

We then move to the analysis of agents' equilibrium bidding strategies. In particular, we focus on the notion of mixed equilibrium. This concept is certainly more appropriate given the anonymity of the players and the fact that each bidder wants to outguess his rivals.[13] However, a proper characterization of the equilibrium appears unfeasible due to the extremely complex nature of the game. In what follows, we are thus only able to

[11] Notice that also in terms of signals agents have no incentives to postpone the submission of their bids. Signals are in fact always up to date and therefore, if anything, an agent may lose some information (the update of the signal) by delaying the submission of his bids.

[12] As a caveat, the full implementation of such a strategy requires $T >> \{\eta_i^{\max}\}_i$, i.e., the auction must last enough as to give players the possibility to submit all their bids if so they wish. We will later confirm (Section 2.3) that in equilibrium the auctioneer indeed sets an optimal duration $T^* >> \{\eta_i^{\max}\}_i$. And also in reality, the condition appears to be always fulfilled: a typical LUBA remains in fact open for a few days while the time a player needs to submit a bid amounts to a few seconds.

[13] Conditional on entry and on players' commitment to submit at most $\eta_i^{\max}$ bids, equilibria in pure strategies may also exist. For instance, if $n = 3$ and $\eta_i^{\max} = 1$ for every $i \in \{a, b, c\}$, the profile $\left\{x_a^1 = 1, x_b^1 = 1, x_c^1 = 2\right\}$ such that c wins or the profile $\left\{x_a^1 = 1, x_b^1 = 2, x_c^1 = 2\right\}$ such that a wins, are Nash equilibria of the bidding phase of the game. No player can in fact increase his payoff by "moving" his bid. We stress that the claim about the existence of pure strategy equilibria holds conditional on players' commitment to $\eta_i^{\max} \geq 1$. Profitable deviations in fact emerge if one relaxes this condition as losing bidders would either submit additional bids (to try to win the auction) or deviate to $\eta_i^{\max} = 0$ (to spare the bidding fees).

describe some qualitative features of the mixed equilibrium that characterizes the bidding phase of a LUBA with signals: in any period an agent places his bid by randomizing according to a probability distribution that is strictly decreasing over a support that is bounded above and whose only possible gaps are determined by the previously submitted bids.

Proposition 1 *At any $t \in \{1, ..., T\}$, a bidder $i \in I$ for which $x_i^t \neq \emptyset$ chooses x_i^t according to the distribution p_i^t defined over the support $S(p_i^t)$ and such that:*
(i) $S(p_i^t) = \left\{\left\{1, ..., \min\left\{\left\{x_i^r - 1|\sigma^{t-1}(x_i^r) = M\right\}_{r=1}^{t-1} \cup \{k_i\}\right\}\right\} \setminus \left\{x_i^r|\sigma^{t-1}(x_i^r) = L\right\}_{r=1}^{t-1}\right\}$ *with* $k_i \leq v_i - c$.
(ii) $p_i^t(x)$ *is strictly decreasing in* x *for* $x \in S(p_i^t)$.

To have an intuition for this result consider first the hypothetical case in which $\eta_i^{\max} = 1$ for every i. This situation would be analogous to a LUBA in which the rules of the auction specify that each player can submit a single bid. Houba *et al.* (2011) and Rapoport *et al.* (2009) carefully investigate such a LUBA. These papers find that in the mixed equilibrium bidders randomize according to a strictly decreasing probability distribution whose support goes from the lowest possible bid up to a certain upper bound $k_i \leq v_i - c$ (any bid $x_i^1 > v_i - c$ is dominated because such a bid would lead to a negative payoff even in case of victory) and with no gaps.[14] The reason is that agents trade-off the chances to pick a low number with the chances to pick a unique number. In equilibrium every number in the support must lead to the same expected payoff. This requires low numbers to be played with higher probability so that they are less likely to result as a unique number. Moreover, the support has no gaps since otherwise bidding on an "empty" number would dominate all the bids above that number and this again cannot happen in equilibrium.

Proposition 1 encompasses this result. In fact, if $\eta_i^{\max} = 1$ for each $i \in I$, signals play no role because at $t = 1$ players have no previous bids and associated signals to condition their behavior on. Therefore, the support of $p_i^1(x)$ simplifies to $S(p_i^1) = \{1, ..., k_i\}$ with

[14] The above mentioned papers actually define the condition $k \leq v - c$ as they assume agents have a common valuation for the auctioned good.

$k_i \leq v_i - c$.[15]

Now consider a bidder with $\eta_i^{\max} > 1$. Given that bids are costly, the agent will place his first bid x_i^1 optimally (i.e., according to $p_i^1(x)$ as defined above). In case he does not conquer the provisional winning bid, the agent will then condition the submission of his additional bids on the updated signals $\sigma^{t-1}(x_i^r)$ associated with his standing bids $\{x_i^r\}_{r=1}^{t-1}$. In particular, not only will the player exclude from the support all the bids that hold the signal L, but he will set the predecessor of the lowest bid that holds the signal M at time $t-1$ (if such a bid exists) as the upper bound of $S(p_i^t)$. In fact, a bid x_i^t placed above (respectively, below) the bound $\min\left\{\{x_i^r - 1|\sigma^{t-1}(x_i^r) = M\}_{r=1}^{t-1}\right\}$ has zero (respectively, positive) probability to receive the signal $\sigma^t\left(x_i^t\right) = W$ (more details appear in the proof in the appendix). Given that agents bid in order to maximize their chances to win and that their payoff is strictly decreasing in the number of submitted bids η_i^T, it follows that the support $S(p_i^t)$ evolves over time as described in Proposition 1.

Notice that the equilibrium defined in Proposition 1 is symmetric in the sense that bidders with the same valuation v_i, as well as with an identical history of bids and signals, employ identical distributions. But bidders who have different v_i and/or who have submitted different bids and/or received different signals randomize according to different distributions.

2.2 The investment decision

Bidders accumulate sunk costs at rate $c > 0$ for every bid they submit. Before the beginning of the bidding phase (i.e., at $t = 0$ when these costs are still prospective), a rational bidder should then set an upper bound on the amount of money he is willing to invest in the game. This bound determines the maximum number of bids the agent can submit. We indicate this number with $\eta_i^{\max} \in \mathbb{N}$. In choosing his maximum level of investment, the bidder must trade-off the probability of winning the LUBA with the losses he suffers in case he does not win. The agent optimally solves this trade-off by maximizing his expected utility $E_i^0(u_i^{T+1})$ at $t = 0$. By rearranging the payoffs defined in Section 2,

[15] We do not explicitly define k_i. We only notice that k_i is certainly a non decreasing function of an agent's private valuation v_i.

the agent's problem can be expressed as:

$$\max_{\eta_i} E_i^0(u_i^{T+1}) = (v_i - \hat{x})\,\phi_i - \eta_i c \tag{1}$$

where ϕ_i indicates the probability that agent i wins the LUBA, i.e., the probability that during the course of the auction player i submits a bid x_i^t such that $\sigma^T\left(x_i^t\right) = W$.

Notice that if $\eta_i^{\max} = \eta^{\max}$ for any $i \in I$ and every agent plays according to the equilibrium strategy defined in Proposition 1, then ex-ante every bidder has the same probability to win the LUBA.[16] On the other hand, a bidder who submits more (respectively less) bids than his opponents has more (less) attempts to try to conquer the winning bid and thus better (worst) chances to win the LUBA.[17] But given that the submission of every bid entails the payment of the fee c, the number of bids that a rational agent is willing to submit in a LUBA depends on the amount of money the player is willing to invest in the game. Call ω_i the monetary investment (or "effort") chosen by agent i. Then, ϕ_i can be expressed as $\phi_i(\omega_1, ..., \omega_n)$. In other words, and still conditional on players bidding according to equilibrium strategies, the ex-ante probability to win the auction depends on the level of investment of player i relative to the levels of investment chosen by the other players.

The bidder's problem defined in (1) can thus be reformulated as:

$$\max_{\omega_i} E_i^0(u_i^{T+1}) = v_i \phi_i(\omega_1, ..., \omega_n) - \omega_i \tag{2}$$

Notice that, with respect to (1), in (2) $\hat{x}$ does not appear. In line with Raviv and

[16] This is analogous to what happens for instance in a one shot matching pennies game: if players are symmetric and they all randomize according to the equilibrium distribution then they are all equally likely to win.

[17] We stress that such a description only applies *in equilibrium*. More in general in fact the probability that player i wins the LUBA depends not only on *how many* bids agent i submits with respect to his opponents but also on *how* the agent (as well as the opponents) submits these bids. For instance, if $\eta_a^{\max} = 6$, $\eta_b^{\max} = 3$, and $\eta_c^{\max} = 4$ then agent a has better chances to win with respect to b and c. But if agent a, stupidly enough, places his six bids on the same number then his chances drop to zero. We are interested in an equilibrium analysis and therefore we consider a context in which every bidder plays according to Proposition 1 (which rules out the use of dominated strategies like the one just mentioned) and knows that his opponents do the same. As such, it is common knowledge that every player places his bids optimally. This implies that the ex-ante probability of winning the LUBA only depends on the "quantity" of the investment and not on its "quality" which is the same for every player.

Virag (2009), we assume in fact that agents, in deciding in $t = 0$ their optimal level of investment ω_i, do not consider that they will also have to pay their winning bid in case they win the LUBA. We already mentioned that in real LUBAs $\hat{x}$ is negligible with respect to the item's market price (around $0.1 - 0.3\%$) and thus unlikely to really affect players' investment decision at $t = 0$.

The maximization problem defined in (2) expresses the investment decision of a LUBA as a rent-seeking game, i.e., a probabilistic contest in which players compete for a prize by expending costly resources.[18] To explicitly solve the problem, we still need to specify a functional form for the success function $\phi_i(\omega_1, ..., \omega_n)$. We tackle this task by first enumerating a number of fundamental properties that must characterize the success function of a LUBA. These properties are:

P1) An agent who does not participate (i.e., he does not submit any bid) has no chances to win.

P2) If there is a unique participant then that bidder wins for sure.

P3) If two bidders invest the same amount of money then their probability of success must be equal.

P4) The probability of winning is weakly increasing in the bidder's level of investment and weakly decreasing in the opponents' one.

More formally:

P1) $\phi_i = 0$ if $\omega_i < c$

P2) $\phi_i = 1$ if $\omega_i \geq c$ and $\omega_j < c$ for all $j \neq i$

P3) $\phi_i = \phi_j$ if $\omega_i = \omega_j$ for any $i, j \in N$

P4) $\frac{\partial \phi_i}{\partial \omega_i} \geq 0$ and $\frac{\partial \phi_i}{\partial \omega_j} \leq 0$ for any $j \neq i$

The simplest specification that satisfies these four properties is the famous Tullock function (Tullock, 1980).[19] We thus employ this specification and set $\phi_i = \frac{\omega_i}{\omega_i + \sum_{j \neq i} \omega_j}$. Problem (2) then becomes:

$$\max_{\omega_i} E_i^0(u_i^{T+1}) = \frac{\omega_i}{\omega_i + \sum_{j \neq i} \omega_j} v_i - \omega_i \tag{3}$$

[18] Rent-seeking games are used to model a wide spectrum of phenomena that involve political lobbying, investment in R&D activities, and lotteries. See Tullock (1980), and Baye and Hoppe (2003).

[19] Skaperdas (1996) actually shows that the Tullock function is the unique success function that obeys a number of basic properties, including the ones we mentioned.

which leads to the following optimal solution:

Proposition 2 *The investment decision of a lowest unique bid auction has solution* $\omega_i^* = \frac{\bar{n}-1}{\bar{n}}\sqrt{\bar{v}v_i} - \left(\frac{\bar{n}-1}{\bar{n}}\right)^2 \bar{v}$.

The optimal level of investment uniquely determines the maximum number of bids that an agent is willing to submit. In fact, introducing the "floor" operator $\lfloor \cdot \rfloor$ such that $\lfloor z \rfloor$ maps the real number z into the integer m with $m \leq z < m+1$, we can state the following proposition.

Proposition 3 *Each bidder submits up to* $\eta_i^{\max} = \left\lfloor \frac{\omega_i^*}{c} \right\rfloor = \left\lfloor \frac{1}{c}\left(\frac{\bar{n}-1}{\bar{n}}\sqrt{\bar{v}v_i} - \left(\frac{\bar{n}-1}{\bar{n}}\right)^2 \bar{v}\right) \right\rfloor$ *bids.*

In line with what intuition suggests, the maximum number of bids that a rational agent is willing to submit in a LUBA is thus a weakly increasing function of the agent's valuation v_i and a weakly decreasing function of the bidding fee c, the average valuation $\bar{v}$, and the expected number of participants $\bar{n}$.

2.3 The auctioneer's decision

Given his outside option $u_a^{T+1} = 0$, the auctioneer's decision of opening the LUBA depends on the expected profits that the mechanism can raise. These are given by:

$$E_a^{-1}\left(\left(\sum_{i=1}^{n} \eta_i^T\right) c + \hat{x} - v_a\right) \tag{4}$$

where the operator $E_a^{-1}(\cdot)$ indicates the auctioneer's expectations at time $t = -1$.

The auctioneer knows n, the number of potential participants, and $\bar{v}$, the mean of the distribution of agents' valuations. He can thus compute the maximum number of bids that a "representative agent" with valuation $\bar{v}$ would submit. This is given by $E_a^{-1}(\eta_i^{\max}) =$

$\left\lfloor \frac{1}{c} \left(\frac{\bar{n}-1}{\bar{n}^2}\right) \bar{v} \right\rfloor$.[20] Moreover, and in line with agents' optimal behavior (see Remark 1 and Proposition1), the auctioneer knows that in equilibrium at least $(n-1)$ agents submit all their available bids. From the auctioneer's point of view the expected profits of a LUBA are thus bounded from below. Proposition 4 formalizes this result while Example 1 explicitly solves the auctioneer's decision in a hypothetical LUBA.

Proposition 4 *A lowest unique bid auction generates expected profits that are strictly larger than* $\underline{u}_a^{T+1} = \left[(n-1) \left\lfloor \frac{1}{c} \left(\frac{\bar{n}-1}{\bar{n}^2}\right) \bar{v} \right\rfloor + 1\right] c - v_a$.

Example 1 *Consider a LUBA for an item for which* $v_r = 10{,}000$ *(i.e., 100 €/\$),* $\bar{v} = 8{,}500$, $c = 100$, *and* $\bar{n} = n = 10$. *These parameters imply* $E_a^{-1}(\omega_i^*) = 765$ *and* $E_a^{-1}(\eta_i^{\max}) = 7$. *Expected profits are then bounded from below by* $\underline{u}_a^{T+1} = (((9 * 7) + 1) * 100) - v_a = 6{,}400 - v_a$. *It follows that* $\underline{u}_a^{T+1} > 0$ *for any* $v_a < 6{,}400$. *The auctioneer thus certainly opens the LUBA if he evaluates the good (or if he manages to acquire it for) less than* 64% *of its retail price.*

Notice that in Example 1 $\bar{n} = n$, i.e., bidders' expectations about the number of participants are correct. More in general, the bound on the expected profitability of a LUBA is an increasing function of the difference $(n - \bar{n})$. For instance, if $\bar{n} = 10$ and $n = 13$ (players underestimate the actual number of participants), minimum profits would have been ensured to be strictly larger than $\underline{u}_a^{T+1} = 8{,}500 - v_a$.[21]

Given the formula for the expected profitability of a LUBA (expression 4 above), one can also easily solve for the optimal amount of the bidding fee c. The fact that bidders can only submit an integer number of bids creates in fact a wedge between a bidder's

[20] The auctioneer expects in fact each bidder $i \in N$ to base his investment decision on the presumption that there are $\bar{n} = \bar{n}_o + 1$ participants with valuation $\bar{v}$ where $\bar{n}_o$ is the mean of the random variable n_o (the number of other players $j \neq i$) . From the auctioneer's point of view bidders are thus involved in a standard Tullock game featuring $\bar{n}$ players with homogeneous valuation $\bar{v}$. This implies $E_a^{-1}(\omega_i^*) = \left(\frac{\bar{n}-1}{\bar{n}^2}\right) \bar{v}$ for any $i \in N$. Because of Proposition 3 this in turn leads to $E_a^{-1}(\eta_i^{\max}) = \left\lfloor \frac{1}{c} \left(\frac{\bar{n}-1}{\bar{n}^2}\right) \bar{v} \right\rfloor$.

[21] This suggests that the auctioneer has incentives in pursuing two simultaneous goals: 1 - increase the number of participants; 2 - let agents believe that there are fewer players than there actually are. Indeed, LUBAs websites put considerable effort in stimulating agents' participation (for instance, by offering special deals to new players, or prizes to bidders that invite their friends to join). They are also pretty transparent about the outcome of closed auctions (for instance, they disclose the winning bid $\hat{x}$ as well as the identity of the winner). But they do not release any information whatsoever about the actual number of participants.

optimal level of investment ω_i^* and his actual investment $\eta_i^{\max}c$ (see propositions 2 and 3). In particular, $\eta_i^{\max}c \leq \omega_i^*$ such that the seller may fail to collect the full willingness to invest of the $(n-1)$ losing bidders. Under the restriction that $c \in \mathbb{N}$ (think of the fee as being defined in €/$ cents), the auctioneer minimizes this loss (and he thus maximizes his profits) by setting the bidding fee c in line with the following proposition.

Proposition 5 *The optimal bidding fee in a LUBA is given by $c^* = 1$.*

In Example 1 above, the setting of $c^* = 1$ raises the lower bound on the auctioneer's expected profits from $\underline{u}_a^{T+1} = 6,400 - v_a$ to $\underline{u}_a^{T+1} = 6,886 - v_a$. In fact, with $c^* = 1$ the nine losing bidders are expected to submit 765 bids such that $E_a^{-1}(\eta_i^{\max})c^* = E_a^{-1}(\omega_i^*) = 765$ and the seller is thus able to collect their entire expected willingness to invest.[22]

In $t = -1$ the auctioneer must also choose and publicly announce the duration of the auction (T). The results about agents' investment decision and bidding behavior imply that the optimal duration T^* is such that $T^* >> E_a^{-1}(\eta_i^{\max})$. The intuition is straightforward: the mechanism generates revenues through the collection of agents' bidding fees; as such, the seller wants the auction to last enough time as to give agents the possibility to use all their $\eta_i^{\max}$ bids if so they wish.

Finally, it is interesting to compare the auctioneer's profits with the profits that the mechanism would raise if signals were not available. The following proposition highlights a result that at first sight may seem surprising.

Proposition 6 *A LUBA with signals is characterized by expected profits that are (weakly) dominated by those that the same mechanism would raise if signals were not used.*

This result leads to question why websites that organize LUBAs implement the mechanism with signals. There are two possible answers: either the auctioneer adopts a sub-optimal behavior or the bidders do not play the game as equilibrium analysis indicates. Given that the first option seems unlikely, we now turn to analyze the second possibility.

[22] While in actual LUBAs one does not observe $c^* = 1$, i.e., a bidding fee of one cent (not ultimately because this would greatly increase the number of bids that agents can submit and there may be costs associated with the management of all these bids and the associated signals), actual fees tend to be pretty low (around 1 Euro/Dollar).

3 The game with (some) boundedly rational bidders

The previous section showed that a lowest unique bid auction can be profitable for the seller even when all the bidders are rational and play the equilibrium strategies. Still, a necessary condition for ensuring positive profits is the existence of sizeable gains from trade. This finding would hardly rationalize the diffusion that websites that organize LUBAs have experienced around the world. On the contrary, this trend seems to suggest that the business, at least in the short period, is much more profitable than what the equilibrium analysis indicates. In this section, we relax the assumption of full rationality and we show how LUBAs can become highly profitable when some bidders lack the necessary commitment to stick to equilibrium strategies.[23] We show how these bidders can get trapped into a costly war of attrition and how this escalation in the level of investments is triggered and amplified by the existence of the signals.

Consider a generic bidder $i \in I$ and let him play according to equilibrium. The agent may be lucky, conquer at some point the provisional winning bid, and eventually win the auction with $\eta_i^T \leq \eta_i^{\max}$. Or, more likely, the player reaches a period $t' < T$ in which he submitted all his bids (i.e., $\eta_i^{t'} = \eta_i^{\max}$) without holding the currently winning bid. In this second case, the agent's provisional payoff at t' is given by $u_i^{t'} = -\eta_i^{\max} c$. A rational bidder who committed to submit at most $\eta_i^{\max}$ bids does not fall victim of the sunk cost fallacy and thus accepts this loss. In other words, he plays $x_i^t = \emptyset$ for any $t \in \{t'+1, ..., T\}$ such that $u_i^{T+1} = -\eta_i^{\max} c$. However, a boundedly rational bidder may be tempted to submit additional bids hoping to eventually win the auction and turn the sunk bidding costs into a positive payoff. We first define what we mean by boundedly rational behavior in the context of a LUBA.

Definition 1 *Bidder i is boundedly rational if:*
(i) whenever $\sigma^t(x_i^r) \neq W$ for every element of the set $\{x_i^r\}_{r=1}^t$, he holds the probability weighting function $w_i^t(q_i^t) > q_i^t \geq 0$, where q_i^t is the probability that an additional bid $x_i^{t+1} \neq \emptyset$ leads to the signal $\sigma^{t+1}(\tilde{x}_i^r) = W$ for a bid $\tilde{x}_i^r \in \{x_i^r\}_{r=1}^{t+1}$.

[23] A similar approach is adopted by Malmendier and Szeidl (2008) who show how the presence of a minority of overbidding behavioral agents disproportionally inflates profits in the case of standard auctions.

(ii) he is myopic and believes that $u_i^{t+1} = u_i^{T+1}$.

(iii) he lacks the commitment to stop at $\eta_i^t = \eta_i^{\max}$.

At any time t, and out of the many possible distributions of actual bids, there are certainly cases in which bidder i can conquer the provisional winning bid by submitting an additional offer. For example, if bidders i and j offered 1 at $t = 1$ then the bid $x_i^2 = 2$ will receive the signal $\sigma^2\left(x_i^2\right) = W$. Therefore, the event of winning the auction with an extra bid has an expected probability $q_i^t \geq 0$. But considering the fact that actual LUBAs usually attract dozens of participants and raise hundreds of bids, this probability, when positive, is certainly small.

In line with prospect theory (Kanheman and Tversky, 1979) and the empirical evidence about probability weighting functions (see Prelec, 1998 and references within), we characterize boundedly rational bidders as those bidders that overestimate this probability, i.e., those who set $w_i^t\left(q_i^t\right) > q_i^t$. Many are the well-known behavioral biases that can shape such a subjective probability assessment: loss-aversion, over-optimism, wishful thinking, bidding fever. The fact that $w_i^t\left(q_i^t\right) > q_i^t$ when q_i^t is small generates a classical pattern of risk attitudes, namely risk-seeking for small probability gains and large probability losses. In the context of a LUBA, this easily translates into excessive bidding.

Proposition 7 *A boundedly rational bidder for which* $\sigma^t\left(x_i^r\right) \neq W$ *for every element of the set* $\left\{x_i^r\right\}_{r=1}^t$ *and* $\eta_i^t \geq \eta_i^{\max}$ *plays* $x_i^{t+1} \neq \emptyset$ *if* $w_i^t\left(q_i^t\right) > \frac{c}{v_i - \hat{x}_i^r} \simeq \frac{c}{v_i}$. *Moreover if this condition holds at time* t *then it also holds at time* $t+k$ *for any* $k \in \{1, ..., T-t\}$.

Example 2 *Consider the situation described in Example 1 with* $v_r = 10,000$, $c = 100$, *and* $\bar{n} = n = 10$. *Let* j *be a boundedly rational bidder with* $v_j = 7,500$ *such that, in line with propositions 2 and 3,* $\omega_j^* = 1461.35$ *and* $\eta_j^{\max} = 14$. *Assume* j *reaches in period* $t' < T$ *the situation* $\eta_j^{t'} = \eta_j^{\max}$ *and* $\sigma^{t'}\left(x_j^r\right) \neq W$ *for any* $x_j^r \in \left\{x_j^1, ..., x_j^{t'}\right\}$. *Proposition 7 states that agent* j *submits an additional bid at period* $t \in \{t'+1, ..., T\}$ *whenever at* $t-1$ *he does not hold the winning bid and his probability weighting function is such that* $w_j^{t'}\left(q_j^{t'}\right) > \frac{100}{7{,}500 - \hat{x}_i^r} \simeq 0.013$. *Notice that the constraint on* $w_j^{t'}\left(q_j^{t'}\right)$ *is very low.*

Example 2 implies that the presence of at least two boundedly rational bidders can easily trigger a costly vicious circle in which these players accumulate sunk costs. A curious feature of the mechanism is that this sort of war of attrition can potentially continue even when the costs associated with the number of submitted bids exceed bidders' valuation of the good on sale and thus agents would realize a negative payoff even if they win the auction. The reason is that bidders compare the sunk costs they already sustained with the potential payoff they would obtain if they win. Even when the latter turns negative still it will be preferable with respect to the former: given that $\hat{x} < v_i$ the relation $v_i - \eta_i^t c - \hat{x} > -\eta_i^t c$ holds in fact at any t.

This feature of lowest unique bid auctions is reminiscent of the Dollar Auction Game (Shubik, 1971).[24] Both in the Dollar Auction Game and in a LUBA, the bidding escalation is detrimental for the bidders but beneficial for the auctioneer. This is not surprising as Morgan and Krishna (1997) show that war of attritions yield revenues that are superior to standard auction mechanisms.

Going back to the analysis of LUBAs, notice that the assumption of two (or more) boundedly rational bidders is not sufficient to trigger the bidding escalation. It is in fact the combination of boundedly rational behavior with the existence of the signals that accomplishes this task. To appreciate the fundamental role that signals play, consider how different the situation would be if agents were not receiving any kind of feedback about the status of their bids. In such a case, each player would hold the legitimate hope to win the auction with one of his $\eta_i^{\max}$ bids such that the incentives to submit extra bids are much weaker. And when at the closure of the auction the winning bid is declared, it would be too late for the losing bidders to submit additional offers. In other words, in terms of ambiguity, a LUBA without signals would resemble a traditional lottery. On the other hand, signals make the game somehow more similar to a "scratch and win" lottery. In fact, signals (and particularly the signal L) immediately inform the bidder that some or all of his offers have no chances to win. This clearly encourages overbidding, given that

[24] The Dollar Auction Game is a public ascending auction where n bidders compete for a dollar. The auction is won by the agent who submits the highest bid but both him and the second highest bidder must pay their bids. Also in this case, the auction is unprofitable for the seller if agents are rational. But if multiple entry occurs, this starts off a bidding war between the two leading bidders such that the winner may end up paying more than what the dollar is worth.

an agent who faces potential losses is tempted to submit additional bids in order to catch up.

Referring to standard auction mechanisms Malmendier and Szeidl (2008) state that "if agents are subject to bidding fever, sellers may instigate this bias using salient messages informing the buyer that he has been outbid". Indeed, the entire signaling mechanism that characterizes LUBAs seems to have been designed with the goal of stimulating emotional responses that may lead to an irrational escalation of commitment. Given that the auctioneer aims to maximize the number of received bids, this obviously comes as no surprise.

4 Conclusions

A lowest unique bid auction (LUBA) is a peculiar selling mechanism that has been experiencing sudden popularity over the Internet. LUBAs allocate valuable goods to the agent who submits the lowest bid that is not matched by any other bid. We showed that if bidders are rational, a LUBA can be profitable for the seller only if there are sizable gains from trade. But we also showed why, in reality, this auction format can be successful: boundedly rational bidders may lack the necessary commitment to stick to equilibrium strategies and they easily get locked in a costly war of attrition that highly rewards the auctioneer. In particular, we highlighted how such a mechanism is driven by the existence of the signals about the current status of players' bids. It is, therefore, ironic to notice how LUBA websites overstress, surely a bit in bad faith, the alleged positive role of these signals.[25] The paper showed instead that signals are, at best, a double-edged weapon.

Lowest unique bid auctions are an ingenious selling mechanism. On one hand, by offering the possibility to win goods of considerable value for very little money, they share the appeal of lotteries. On the other hand, they give bidders the illusion of being in control of what they do, and they convey the idea that winning is just a matter of being smarter than the others. The combination of these two factors makes the business successful, at

[25] For instance, one of these websites claims that "Relying on these signals, using different strategies and different levels of investment, to win the auction becomes a matter of a complex use of various abilities". Another website declares: "The investment, the signals and the bidding strategies make the auction void of any element of luck and based exclusively on the bidder's ability".

least in the short run: early entrants can in fact exploit the enthusiasm and the naivete of (boundedly rational) consumers. We conclude by stressing once more the similarities that lowest unique bid auctions have with other well-known games such as the war of attrition and the Dollar Auction Game. It is obviously not a coincidence that these games are used as archetypes for describing situations where irrational behavior leads to an over-investment of resources.

5 Appendix

Proof of Proposition 1

We divide the proof in two steps. In 1) we define the equilibrium for the case in which $\eta_i^{\max} = 1$ for every $i \in I$. This part essentially replicates the proofs in Houba *et al.* (2011) and Rapoport *et al.* (2009). In 2) we prove the result for the more generic case with $\eta_i^{\max} \geq 1$ for every $i \in I$.

1) $\eta_i^{\max} = 1$ for every $i \in I$.

If $\eta_i^{\max} = 1$ for every $i \in I$ signals play no role. In fact, in deciding where to submit their bid x_i^1, players have no standing bids and associated signals to condition their behavior on. We thus have to show that the support of p_i^1 is given by $S(p_i^1) = \{1, ..., k_i\}$ with $k_i \leq v_i - c$ and that $p_i^1(x)$ is strictly decreasing in x with $x \in S(p_i^1)$.

It is immediate to show that the support $S(p_i^t)$ is bounded above by some number $k_i \leq v_i - c$. Any bid $x_i^1 = \delta$ with $\delta > v_i - c$ is in fact dominated. Such a bid would lead to a negative payoff no matter if it turns out to be a losing bid ($u_i^{T+1} = -c$) or the winning bid ($u_i^{T+1} = v_i - c - \delta < 0$).

We then prove that $p_i^1(x)$ is strictly decreasing in x. Let $\lambda, \omega \in S(p_i^1)$ with $\min\left\{p_i^1(\lambda), p_i^1(\omega)\right\} > 0$ and assume without loss of generality that $\lambda < \omega$. By definition of symmetric mixed strategy Nash equilibrium, the fact that both $x_i^1 = \lambda$ and $x_i^1 = \omega$ are played with positive probability implies that, whenever each agent plays according to the distribution $p_i^1(x)$, then the following condition must hold:

$$E_i^0\left(u_i^{T+1} | x_i^1 = \lambda\right) = E_i^0\left(u_i^{T+1} | x_i^1 = \omega\right) \tag{5}$$

i.e., a player must be indifferent between placing his bid on λ or ω. Now assume that $p_i^1(\lambda) \leq p_i^1(\omega)$. This would actually imply that $E_i^0\left(u_i^{T+1} | x_i^1 = \lambda\right) > E_i^0\left(u_i^{T+1} | x_i^1 = \omega\right)$ and thus contradicts condition (5). In fact, if $p_i^1(\lambda) \leq p_i^1(\omega)$ the probability of $x_i^1 = \lambda$ being unique would be larger or equal than the one of $x_i^1 = \omega$. This implies that $x_i^1 = \lambda$ would be more likely to result into the *lowest* unique bid given that $\lambda < \omega$. Moreover with $x_i^1 = \lambda$ the price that the bidder pays if he wins would be lower than with $x_i^1 = \omega$.

Therefore, for (5) to hold it must be the case that $p_i^1(\omega) < p_i^1(\lambda)$. In other words, in equilibrium the bid $x_i^1 = \lambda$ must have lower chances to be unique with respect to $x_i^1 = \omega$ (i.e., it must be played with a higher probability) in order to balance the advantages that stem from the fact that $\lambda < \omega$. By setting $\omega = \lambda + 1$, the result that $p_i^1(\omega) < p_i^1(\lambda)$ implies that the support of $p_i^1(x)$ is given by $S(p_i^1) = \{1, ..., k_i\}$, i.e., it starts from 1 and has no gaps.

2) $\eta_i^{\max} \geq 1$ for every $i \in I$.

Consider the situation of an agent that at time $t > 1$ already submitted the bids $\{x_i^r \neq \emptyset\}_{r=1}^{t-1}$. Assume that the agent does not hold the currently winning bid and let $\eta_i^{t-1} < \eta_i^{\max}$. In line with Remark 1 the agent thus decides to submit an additional bid $x_i^t \neq \emptyset$. Given that bids are costly, the agent's goal is to conquer the winning bid with the submission of this marginal bid. In deciding where to place x_i^t the agent has the following options:

a) submit $x_i^t = \tilde{x}_i^r$ with $\tilde{x}_i^r \in \{x_i^r \neq \emptyset\}_{r=1}^{t-1}$.

b) submit $x_i^t \neq \tilde{x}_i^r$ for any $\tilde{x}_i^r \in \{x_i^r \neq \emptyset\}_{r=1}^{t-1}$ and such that $x_i^t > \min\left\{\tilde{x}_i^r | \sigma^{t-1}\left(\tilde{x}_i^r\right) = M\right\}$.

c) submit $x_i^t \neq \tilde{x}_i^r$ for any $\tilde{x}_i^r \in \{x_i^r \neq \emptyset\}_{r=1}^{t-1}$ and such that $x_i^t < \min\left\{\tilde{x}_i^r | \sigma^{t-1}\left(\tilde{x}_i^r\right) = M\right\}$.

Option a (i.e., replicate a bid that has already been submitted) has clearly no chances to conquer the winning bid. In particular $\Pr\left[\sigma^t\left(x_i^t\right) = L\right] = 1$ which obviously implies $\Pr\left[\sigma^t\left(x_i^t\right) = W\right] = 0$.

Options b and c actually coincide whenever the set $\min\left\{\tilde{x}_i^r | \sigma^{t-1}\left(\tilde{x}_i^r\right) = M\right\}$ is empty, i.e., none of the agent's standing bids holds the signal M. But they do differ when this is not the case. To discriminate between the two options thus assume that the set $\min\left\{\tilde{x}_i^r | \sigma^{t-1}\left(\tilde{x}_i^r\right) = M\right\}$ contains at least one element.

The marginal bid x_i^t has no possibility to receive the signal W in t if the agent bids according to option b. In particular: $\Pr\left[\sigma^t\left(x_i^t\right) = W\right] = 0$ (this follows directly from the fact that the player already submitted a bid $\tilde{x}_i^r < x_i^t$ that holds the signal M), $\Pr\left[\sigma^t\left(x_i^t\right) = L\right] > 0$, and $\Pr\left[\sigma^t\left(x_i^t\right) = M\right] > 0$. Notice that if the event $\sigma^t\left(x_i^t\right) = M$ occurs then x_i^t can eventually receive the signal W in a subsequent period $t' > t$. But in the meantime, and in line with Remark 1 and the equilibrium bidding strategy, agent i will submit additional bids (as far as $\eta_i^{t-1} < \eta_i^{\max}$) and thus incur into higher costs and experience a lower payoff no matter if he wins or if he loses the LUBA.

An agent that submits his bid according to option c has instead a positive probability of conquering the currently winning bid. More precisely: $\Pr\left[\sigma^t\left(x_i^t\right)=W\right]>0$ (again this directly follows from the fact that the player already submitted a bid $\tilde{x}_i^r<x_i^t$ that holds the signal M), $\Pr\left[\sigma^t\left(x_i^t\right)=L\right]>0$, and $\Pr\left[\sigma^t\left(x_i^t\right)=M\right]>0$. Notice moreover that even if the event $\sigma^t\left(x_i^t\right)=L$ occurs, the submission of x_i^t can nevertheless generate the signal W on agent i's standing bid $\min\left\{\tilde{x}_i^r|\sigma^{t-1}\left(\tilde{x}_i^r\right)=M\right\}$. For this to happen two conditions must verify: I) x_i^t matches $\hat{x}_j^r$ where $\hat{x}_j^r$ was the provisional winning bid placed by player $j\neq i$ (i.e., $\sigma^{t-1}\left(\hat{x}_j^r\right)=W$); and II) the current distribution of bids is such that the bid $\min\left\{\tilde{x}_i^r|\sigma^{t-1}\left(\tilde{x}_i^r\right)=M\right\}$ becomes the new lowest unique bid.

Option c thus dominates the alternatives. Therefore, a bidder must place the marginal bid x_i^t according to a probability distribution p_i^t whose support excludes all the previously submitted bids $\{x_i^r\neq\emptyset\}_{r=1}^{t-1}$ and is bounded above by the predecessor of the smallest bid that holds the signal $\sigma^{t-1}\left(x_i^r\right)=M$ (if such a bid exists). In other words, the support $S(p_i^t)$ is such that:

$$S(p_i^t)=\left\{\left\{1,...,\min\left\{\left\{x_i^r-1|\sigma^{t-1}\left(x_i^r\right)=M\right\}_{r=1}^{t-1}\cup\{k_i\}\right\}\right\}\setminus\left\{x_i^r|\sigma^{t-1}\left(x_i^r\right)=L\right\}_{r=1}^{t-1}\right\}$$

Where k_i is such that $k_i\leq v_i-c$ (see point 1 in this proof). The proof that $p_i^t(x)$ is strictly decreasing over $S(p_i^t)$ replicates the proof in 1).

Proof of Proposition 2

Generic agent i solves:

$$\max_{\omega_i}E_i^0\left(u_i^{T+1}\right)=\left(\frac{\omega_i}{\omega_i+\sum_{j\neq i}\omega_j}\right)v_i-\omega_i \tag{6}$$

The agent knows his private valuation (v_i) but he does not know the valuations of the other participants (v_j with $j\neq i$). Therefore, the agent does not know, nor he can infer, their level of investment ($\omega_j(v_j)$). Agent i only knows the mean of the distribution of agents' valuations ($\bar{v}$) and therefore he sets $\omega_j(v_j)=\omega_j(\bar{v})$ for any $j\neq i$ (see Gallice, 2013, for more details on the procedure). The agent also faces uncertainty concerning the actual number of other players (n_o). From his point of view n_o is in fact a random variable distributed on $\{0, ..., n_{\max}\}$ according to the distribution g. In tackling his investment decision, the agent thus considers the expected value $\bar{n}_o = \sum_{n_o=1}^{n_{\max}} n_o g(n_o)$. The problem then becomes:

$$\max_{\omega_i} E_i^0 \left(u_i^{T+1}\right) = \left(\frac{\omega_i}{\omega_i + \bar{n}_o \omega_j(\bar{v})}\right) v_i - \omega_i \tag{7}$$

Consistently with the above mentioned hypothesis, agent i expects that in equilibrium $\omega_j^*(\bar{v}) = \left(\frac{\bar{n}_o}{(\bar{n}_o+1)^2}\right) \bar{v} = \left(\frac{\bar{n}-1}{\bar{n}^2}\right) \bar{v}$ where $\bar{n} = \bar{n}_o + 1$ is the expected total number of participants in the LUBA. Notice in fact that not only player i assigns a valuation $\bar{v}$ to any agent $j \neq i$; he also expects any $j \neq i$ to adopt a similar behavior, i.e., he expects any $j \neq i$ to assign a valuation $\bar{v}$ to any of his $\bar{n}_o$ opponents (thus including agent i). From agent's i point of view, every agent $j \neq i$ is thus involved in a Tullock game featuring $\bar{n} = \bar{n}_o + 1$ players with homogeneous valuation $\bar{v}$. The problem is thus characterized by the standard solution $\omega_j^*(\bar{v}) = \left(\frac{\bar{n}-1}{\bar{n}^2}\right) \bar{v}$.

It follows that agent's i problem becomes:

$$\max_{\omega_i} E_i^0 \left(u_i^{T+1}\right) = \left(\frac{\omega_i}{\omega_i + (\bar{n}-1)\left(\frac{\bar{n}-1}{\bar{n}^2}\right) \bar{v}}\right) v_i - \omega_i \tag{8}$$

Necessary and sufficient conditions are:

$$\frac{\partial E_i^0 \left(u_i^{T+1}\right)}{\partial \omega_i} = \left(\frac{\left(\frac{\bar{n}-1}{\bar{n}}\right)^2 \bar{v}}{\left(\omega_i + \left(\frac{\bar{n}-1}{\bar{n}}\right)^2 \bar{v}\right)^2}\right) v_i - 1 = 0 \tag{9}$$

$$\frac{\partial^2 E_i^0 \left(u_i^{T+1}\right)}{\partial \omega_i^2} = -\frac{2\left(\frac{\bar{n}-1}{\bar{n}}\right)^2 \bar{v}}{\left(\omega_i + \left(\frac{\bar{n}-1}{\bar{n}}\right)^2 \bar{v}\right)^3} v_i < 0 \tag{10}$$

which lead to the optimal solution:

$$\omega_i^* = \frac{\bar{n}-1}{\bar{n}} \sqrt{\bar{v} v_i} - \left(\frac{\bar{n}-1}{\bar{n}}\right)^2 \bar{v} \tag{11}$$

Proof of Proposition 3

Agent i is willing to invest up to ω_i^* and each bid costs c. Therefore, $\frac{\omega_i^*}{c}$ is the maximum number of bids agent i would submit if $\eta_i^{\max} \in \mathbb{R}$. But given that $\eta_i^{\max} \in \mathbb{N}$ and ω_i^* is an upper bound, it follows that $\eta_i^{\max} = \left\lfloor \frac{\omega_i^*}{c} \right\rfloor$.

Proof of Proposition 4

Given that

$$u_a^{T+1} = \begin{cases} \left(\sum_{i=1}^{n} \eta_i^T\right) c + \hat{x} - v_a & \text{if } a \text{ opens the auction} \\ 0 & \text{otherwise} \end{cases}$$

in $t = -1$ the auctioneer decides to open the auction if this action leads to a positive expected payoff, i.e., if $E_a^{-1}\left(\left(\sum_{i=1}^{n} \eta_i^T\right) c + \hat{x} - v_a\right) > 0$. In equilibrium the $n-1$ losing bidders submit $\eta_i^{\max}$ bids while the winning bidder submits at least one bid. Moreover the winning bid $\hat{x}$ is such that $\hat{x} \geq 1$. It follows that:

$$E_a^{-1}\left(\left(\sum_{i=1}^{n} \eta_i^T\right) c + \hat{x} - v_a\right) \geq \left((n-1)E_a^{-1}\left(\eta_i^{\max}\right) + 1\right) c + 1 - v_a \tag{12}$$

and thus:

$$E_a^{-1}\left(\left(\sum_{i=1}^{n} \eta_i^T\right) c + \hat{x} - v_a\right) > \left((n-1)E_a^{-1}\left(\eta_i^{\max}\right) + 1\right) c - v_a \tag{13}$$

Because of Proposition 3, $E_a^{-1}\left(\eta_i^{\max}\right) = \left\lfloor \frac{1}{c}\left(\frac{\bar{n}-1}{\bar{n}^2}\right)\bar{v}\right\rfloor$ given that the auctioneer considers $v_i = \bar{v}$ for every $i \in N$. Therefore:

$$E_a^{-1}\left(\left(\sum_{i=1}^{n} \eta_i^T\right) c + \hat{x} - v_a\right) > \left[(n-1)\left\lfloor \frac{1}{c}\left(\frac{\bar{n}-1}{\bar{n}^2}\right)\bar{v}\right\rfloor + 1\right] c - v_a \tag{14}$$

Proof of Proposition 5

Given the investment decision $\omega_i^* \in \mathbb{R}$ of the generic (losing) bidder i, the auctioneer wants to choose $c^* \in \mathbb{N}$ in order to maximize the amount $\eta_i^{\max} c = \left\lfloor \frac{\omega_i^*}{c} \right\rfloor c$ that agent i pays in equilibrium (see Proposition 3). Decompose ω_i^* such that $\omega_i^* = \lfloor \omega_i^* \rfloor + \{\omega_i^*\}$ where $\{\omega_i^*\} \in [0,1)$ is the fractional part of ω_i^*. Then $\left\lfloor \frac{\omega_i^*}{c} \right\rfloor c = \left\lfloor \frac{\lfloor \omega_i^* \rfloor}{c} + \frac{\{\omega_i^*\}}{c} \right\rfloor c$. Now assume that c is a divisor of $\lfloor\omega_i^*\rfloor$, i.e., $\lfloor\omega_i^*\rfloor = mc$ with $m \in \mathbb{N}$. Then $\left\lfloor \frac{\lfloor\omega_i^*\rfloor}{c} + \frac{\{\omega_i^*\}}{c} \right\rfloor = m$ given that $\frac{\lfloor\omega_i^*\rfloor}{c} = m$ and $\frac{\{\omega_i^*\}}{c} \in [0,1)$. It follows that $\left\lfloor \frac{\lfloor\omega_i^*\rfloor}{c} + \frac{\{\omega_i^*\}}{c} \right\rfloor c = mc = \lfloor\omega_i^*\rfloor$. With respect to the first-best ω_i^*, the auctioneer thus fails to collect only the fractional part $\{\omega_i^*\}$. Consider instead the case in which c is not a divisor of $\lfloor\omega_i^*\rfloor$, i.e., $\lfloor\omega_i^*\rfloor = m'c$ with $m' \notin \mathbb{N}$. Then $\left\lfloor \frac{\lfloor\omega_i^*\rfloor}{c} + \frac{\{\omega_i^*\}}{c} \right\rfloor < m'$ given that $\lfloor m' \rfloor < \left(\frac{\lfloor\omega_i^*\rfloor}{c} + \frac{\{\omega_i^*\}}{c} \right) < \lfloor m' \rfloor + 1$ and $m' \notin \mathbb{N}$. It follows that $\left\lfloor \frac{\lfloor\omega_i^*\rfloor}{c} + \frac{\{\omega_i^*\}}{c} \right\rfloor c < m'c = \lfloor\omega_i^*\rfloor$. In such a case, the auctioneer thus fails to collect strictly more than just the fractional part $\{\omega_i^*\}$. It is therefore dominant for the auctioneer to choose a fee c that is a divisor of $\lfloor\omega_i^*\rfloor$. Obviously, any given $\lfloor\omega_i^*\rfloor$ may have multiple divisors. But given that agents are heterogeneous in ω_i^* and that the auctioneer does not know these values, the optimal bidding fee c^* is the one that is ensured to be a divisor of every number in the set $\{\lfloor\omega_i^*\rfloor\}_{i=1}^n$, i.e., $c^* = 1$.

Proof of Proposition 6

The expected profitability of a lowest unique bid auction with signals is given by:

$$E_a^{-1}\left(u_a^{T+1}|signals\right) = \left((n-1)E_a^{-1}(\eta_i^{\max}) + E_a^{-1}\left(\hat{\eta}_i^T\right)\right)c + \hat{x} - v_a \tag{15}$$

where $\hat{\eta}_i^T \in \left\{1, ..., E_a^{-1}(\eta_i^{\max})\right\}$ is the number of bids submitted by the winning bidder. If signals were banned then auctioneer's expected profits would instead amount to:

$$E_a^{-1}\left(u_a^{T+1}|nosignals\right) = nE_a^{-1}(\eta_i^{\max})c + \hat{x} - v_a \tag{16}$$

because with no feedbacks all the n bidders would submit their $\eta_i^{\max}$ available bids. Notice that, for any given $\hat{x}$, (14) = (15) if and only if $\hat{\eta}_i^T = \eta_i^{\max}$, i.e., the winning bidder conquers the winning bid on his last attempt. In all other cases, we have (14) < (15). We thus can conclude that (14) $\leq$ (15), i.e., the expected profits of a LUBA with signals are weakly dominated by the expected profits of a LUBA that employs no signals.

Proof of Proposition 7

A boundedly rational bidder who does not hold the winning bid and is not committed to $\eta_i^t \leq \eta^{\max}$, submits an additional bid if $E_i^t(u_i^{t+1}) > u_i^t$, i.e., if the following condition holds:

$$w_i^t\left(q_i^t\right)\left(v_i - \left(\eta_i^t + 1\right)c - \hat{x}^r\right) + \left(1 - w_i^t\left(q_i^t\right)\right)\left(-\left(\eta_i^t + 1\right)c\right) > -\eta_i^t c \tag{17}$$

where $\hat{x}^r \in \{x_i^r\}_{r=1}^{t+1}$ is i's eventual provisional winning bid at period $t+1$. Solving for $w_i^t\left(q_i^t\right)$, the condition holds for any $w_i^t\left(q_i^t\right) > \frac{c}{v_i - \hat{x}^r} \simeq \frac{c}{v_i}$ given that $\hat{x}^r$ is negligible. With this approximation the lower bound for the probability weighting function does not depend on η_i^t and remains constant over time. This means that if the constraint is satisfied at period t, it is also satisfied at any period $t+k$ with $k \in \{1, ..., T-t\}$.

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