Bankruptcy Problems with Reference-Dependent Preferences

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Abstract

I study bankruptcy problems under the assumption that claimants have reference-dependent preferences. I consider different specifications for claimants' reference points and show how perceived gains and losses impact on aggregate welfare. I can thus rank the four most prominent rules (Proportional, Constrained Equal Awards, Constrained Equal Losses, and Talmud) on the basis of the level of utilitarian and maxmin welfare that they generate. When none of these rules maximizes welfare, I identify the rule that does it and discuss its properties.

Keywords: bankruptcy problems, reference-dependent preferences, reference points, utilitarian welfare, maxmin welfare.

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1. Introduction

In a bankruptcy problem, an arbitrator must allocate a finite and perfectly divisible resource among several claimants whose claims sum up to a greater amount than what is available. Situations that match this description include the liquidation of a bankrupted firm among its creditors, the division of an estate among heirs, or the allocation of time to the completion of projects assigned by different clients.

The formal analysis of bankruptcy problems started with O'Neill (1982) and has flourished since that time (see Moulin, 2002 and Thomson, 2003, 2015 for detailed surveys). The research question that underlies the literature is as follows: how shall the arbitrator adjudicate conflicting claims? The answer usually takes the form of an allocation rule, i.e., a procedure that processes the data of the problem (namely, the endowment of the resource and the list of individual claims) and then prescribes an allocation for the arbitrator to implement. The analysis is pursued under the assumption that claimants have monotonically increasing preferences. However, the specific functional form of these preferences is usually left unspecified. As Thomson puts it (2015, p. 57): "In the base model, preferences are not explicitly indicated, but it is implicit that each claimant prefers more of the dividend to less".

In this paper, I study bankruptcy problems when claimants' preferences have an explicit formulation. More precisely, I consider the case of reference-dependent preferences (RDPs), as introduced by Kőszegi and Rabin (2006). Building upon the main insights of prospect theory (Kahneman and Tversky, 1979), RDPs acknowledge the fact that an agent's perception of a given outcome is determined not only by the outcome per se but also by how this outcome compares with a certain reference point. In other words, the agent's utility is influenced by perceived gains and losses. RDPs thus seem particularly appropriate for use in capturing the preferences of claimants in bankruptcy problems. These are, in fact, typical situations in which agents form expectations about what they will get and then inevitably compare the actual outcome with the expected one.

The idea that reference points might play a role in bankruptcy problems is not new. Chun and Thomson (1992) studied a bargaining problem with claims and interpret the disagreement point as a reference point from which agents measure their gains when evaluating a proposal. Herrero (1998) adopts a similar framework but endogenizes the reference point as a function of the agents' claims and the set of feasible allocations. Pulido et al. (2002, 2008) study bankruptcy problems with reference points in the context of university budgeting procedures. Finally, Hougaard et al. (2012, 2013a, 2013b) consider a more general model of rationing in which agents have claims as well as baselines, which can also be interpreted as reference points. All these papers, however, analyze the role of reference points in a context in which claimants have standard preferences.

I instead embed the analysis of reference points into the framework of RDPs and focus on the welfare implications that such a setting generates. I consider different specifications for claimants' reference points. This reflects the role that expectations have in determining reference points (Abeler et al., 2011; Ericson and Fuster, 2011) and the fact that in a bankruptcy problem there are multiple allocations that can catalyze claimants' expectations. I thus let the vector that collects agents' reference points to coincide with the claims vector, the zero awards vector, the minimal rights vector, and with their beliefs about the awards vector that the arbitrator will implement.

The actual feasibility of these reference points paired with some specific features of RDPs impact on how different rules perform in terms of (utilitarian and maxmin) welfare. For instance, claimants' diminishing marginal sensitivity to losses implies that, whenever reference points are not feasible, the rules that achieve higher welfare are those that most asymmetrically allocate perceived losses across claimants. In the opposite scenario, diminishing marginal sensitivity to gains implies that, when agents' reference points are mutually feasible, the best rules are those that implement the most equal distributions of perceived gains.

I can thus rank the four most common rules (Proportional, Constrained Equal Awards, Constrained Equal Losses, and Talmud) for any specification of agents' reference points. The Constrained Equal Awards rule often outperforms other rules. It may, however, fail to select the first-best allocation. When this is the case, I define the rule that maximizes welfare and discuss its properties. For instance, I fully characterize the Small Claims First (SCF) rule, which is the rule that maximizes utilitarian welfare when claimants use their claims as reference points. The Minimal Utility Gap (*MUG*) rule instead maximizes maxmin welfare when reference points are given by agents' claims or by their minimal rights. And the Constrained Equal Gains (*CEG*) rule is optimal when agents use as reference points their minimal rights and the arbitrator cares about maxmin welfare. In the course of the analysis, I also discuss the issue of duality and the possibility that claimants have heterogeneous gain-loss functions.

More generally, the analysis highlights the existence of a trade-off between the goal of welfare maximization and the equity of the resulting award vector. This is most evident when claimants use their claims as reference points, as in this case at least some of the agents must necessarily receive less than what they were expecting. The optimal rule (the *SCF* rule) then prescribes the arbitrator to satisfy as many claimants as possible (i.e., to allocate them what they claim) while simultaneously disappointing at the most those who can be disappointed the most. I show that the *SCF* rule fails *Equal Treatment of Equals*, although it satisfies a weaker notion of equity, as embedded in a property that I label *Ex-Ante Equal Treatment of Equals*. The tension between welfare maximization and equity gets further exacerbated if one is willing to give up the property of *Boundedness*, which forbids an arbitrator to allocate to any claimant more than his claim. The fact that, because of RDPs, losses loom larger than gains implies that in some circumstances this is exactly what the arbitrator should do. This would, however, lead to an even more skewed distribution of the endowment. In particular, it may hinder some of the claimants from obtaining their minimal rights. The resulting allocation would then be perceived as extremely unfair by some of the agents.

2. The Model

2.1 A Bankruptcy Problem

Let $E \in \mathbb{R}_+$ denote the endowment of the resource to be allocated and $N = \{1, ..., n\}$ be the set of claimants. Each claimant $i \in N$ has a claim $c_i \in \mathbb{R}_+$ on E. The vector $c = (c_1, ..., c_n)$ with $\sum_i c_i = C$ collects individual claims. A *bankruptcy problem* (or claims problem) is a pair $(c, E) \in \mathbb{R}_+^N \times \mathbb{R}_+$ where c is such that $C \geq E$. I denote with $\mathbb{B}^N$ the class of all such problems. Note that by defining as $L = C - E$ the aggregate loss, the problem (c, L) is the dual of the problem (c, E). In other words, one can interpret a bankruptcy problem as a problem of allocating what is available (i.e., shares of E), or as a problem of allocating what is missing (i.e., shares of L).[1]

A *rule* R is a function that associates to any problem $(c, E) \in \mathbb{B}^N$ a unique awards vector $R(c, E) = (R_1(c, E), ..., R_n(c, E))$. The awards vector $R(c, E)$ must satisfy the following two conditions:

(i) $0 \leq R_i(c, E) \leq c_i$ for any $i \in N$ (*Boundedness*)

(ii) $\sum_{i \in N} R_i(c, E) = E$ (*Balance*)

The literature has characterized a large number of rules that respond to different ethical or procedural criteria (see Thomson, 2015, for a review). In what follows, I focus on the four most prominent rules (Herrero and Villar, 2001; Bosmans and Lauwers, 2011):

- The *Proportional* (P) rule, which allocates the endowment proportional to claims:

$$P(c, E) = \lambda c \text{ with } \lambda = E/C$$

- The *Constrained Equal Awards* (CEA) rule, which assigns equal awards to all claimants subject to the requirement that no one receives more than his claim:

$$CEA_i(c, E) = \min\{c_i, \lambda\} \text{ for all } i \in N \text{ with } \sum_i \min\{c_i, \lambda\} = E$$

- The *Constrained Equal Losses* (CEL) rule, which assigns an equal amount of losses to all claimants subject to the requirement that no one receives a negative amount:

$$CEL_i(c, E) = \max\{0, c_i - \lambda\} \text{ for all } i \in N \text{ with } \sum_i \max\{0, c_i - \lambda\} = E$$

[1]The (c, L) formulation is particularly appropriate when the problem consists in allocating tax burdens, as the vector c can be thought as collecting agents' gross incomes and L is the tax to be levied (Young, 1988; Chambers and Moreno-Ternero, 2017), or in deciding how to finance a public good, as c can describe agents' benefits from the usage of the good and L is the cost to be shared (Moulin, 1987).

- The *Talmud* (T) rule, as introduced by Aumann and Maschler (1985), which foresees two different solutions depending upon the relationship between the half-sum of the claims and the endowment:

$$T(c,E) = CEA\left(\frac{1}{2}c, E\right) \text{ if } \frac{C}{2} \geq E$$

$$T(c,E) = \frac{1}{2}c + CEL\left(\frac{1}{2}c, E - \frac{C}{2}\right) \text{ if } \frac{C}{2} < E$$

Example 1 shows how the four rules works in practice. The example also illustrates some duality results. Two rules R and R^* are dual if $R(c,E) = c - R^*(c,L)$. Then, the Proportional rule and the Talmud rule are self-dual, whereas the Constrained Equal Awards rule and the Constrained Equal Losses rule are dual of each other.

EXAMPLE 1. *Consider the problem* (c,E) *with* $c = (30,50,80)$ *and* $E = 100$. *The four rules select the awards vectors* $P(c,E) = (18.75, 31.25, 50)$, $CEA(c,E) = (30,35,35)$, $CEL(c,E) = (10,30,60)$, *and* $T(c,E) = (15, 27.5, 57.5)$.
In the dual problem (c,L) *with* $L = 60$ *the rules instead select* $P(c,L) = (11.25, 18.75, 30)$, $CEA(c,L) = (20,20,20)$, $CEL(c,L) = (0,15,45)$, *and* $T(c,L) = (15, 22.5, 22.5)$.

2.2 Claimants' Preferences and Social Welfare

I deviate from the baseline model of a bankruptcy problem by assuming that claimants have reference-dependent preferences. I adopt the specification originally proposed by Kőszegi and Rabin (2006) (see Shalev, 2000, for an alternative approach) and thus endow claimants with the following utility function:

$$u(R_i(c,E) \mid r_i) = R_i(c,E) + \mu(R_i(c,E) - r_i) \tag{1}$$

where, as before, $R_i(c,E)$ is the amount that agent i receives from the arbitrator when the latter uses rule R and $r_i \in [0, c_i]$ is the agent's reference point, whose nature I will shortly discuss. The utility that the agent enjoys from the possession/consumption of $R_i(c,E)$ is thus linear, as it is usually assumed in the baseline model.[2] However, the agent's overall utility is now also influenced by the function $\mu(\cdot)$ which captures the additional effects of perceived gains and losses with respect to the reference point. In line with the original formulation of prospect theory (Kahneman and Tversky, 1979), the function $\mu(\cdot)$ is assumed to be continuous, strictly increasing and such that $\mu(0) = 0$. It is is strictly convex in the domain of losses ($\mu''(z) > 0$ for any $z < 0$) and strictly concave in the domain of gains ($\mu''(z) > 0$ for any $z < 0$). Finally, losses loom larger than gains: $|\mu(-z)| > \mu(z)$ for any $z > 0$.[3]

As measures of welfare, I rely on the two most common notions (see Moulin, 2003, and Gravel and Moyes, 2013): utilitarian welfare and maxmin welfare. Utilitarian welfare amounts to the sum of individual utilities. Rule R thus generates utilitarian welfare:

$$W_{ut}(R) = \sum_i (R_i(c,E) + \mu(R_i(c,E) - r_i)) = E + \sum_i \mu(R_i(c,E) - r_i) \tag{2}$$

where the condition $\sum_i R_i(c,E) = E$ holds because of *Balance*. Maxmin (or Rawlsian) welfare is instead defined by the well-being of the worst-off individual. Rule R thus achieves maxmin welfare:

$$W_{mm}(R) = \min\{R_i(c,E) + \mu(R_i(c,E) - r_i)\}_{i \in N} \tag{3}$$

Both measures of welfare thus explicitly take into account the "behavioral" part of claimants' utility functions, namely the gain-loss function $\mu(\cdot)$. The approach is in line with the recent literature on behavioral welfare economics (see Bernheim and Rangel, 2007 and 2009, and Fleurbaey and Schokkaert, 2013, for a general discussion of the issue; see Gruber and Kőszegi, 2004, and O'Donoghue and Rabin, 2006, for more specific applications) and fits the situations that motivate the paper. For instance, a politician who must

[2]Thomson (2015, p. 57) writes that the baseline model amounts to "... assuming that the utilities that claimants derive from their assignments are linear, or to ignoring utilities altogether". Exceptions to this approach include Mariotti and Villar (2005) and Herrero and Villar (2010) that explicitly set up the problem in a utility space.

[3]The properties of $\mu(\cdot)$ drive most of the results in the paper. Indeed, the alternative utility specification $u(R_i(c,E) \mid r_i) = \mu(R_i(c,E) - r_i)$ would lead to similar insights. I use (1) because it explicitly disentangles consumption utility from the utility that stems from perceived gains and losses (see Kőszegi and Rabin, 2006 and 2007). I consider the possibility of heterogeneity in $\mu(\cdot)$ in Section 4.3.

distribute a limited amount of public funds to different associations and cares about his chances of being reelected will certainly take into account how different allocations impact on claimants' degree of satisfaction/disappointment. Similarly, an agent who must allocate his time to the completion of different projects and cares about future collaborations with his clients must carefully consider which are the ones to please and the ones to disappoint.

Clearly, the four standard rules are welfare equivalent when all agents have identical claims ($c_i = c_j$ for all $i, j \in N$). All rules in fact select the egalitarian allocation, $R_i(c, E) = E/n$ for all $i \in N$. Therefore, in what follows I mainly focus on the more interesting case in which claimants are asymmetric, i.e., the vector of claims c is such that $c_i \neq c_j$ for some $i, j \in N$.

3. Reference Points

Claimants' utility function is given by equation (1). Here I discuss the nature of agents' reference points $r = (r_1, ..., r_n)$. I consider different specifications for r. In Section 3.1, I study the case in which agents' reference points coincide with their claims, $r = c$. In Section 3.2, I consider the opposite case where agents set as reference points the zero awards vector, $r = \mathbf{0}$. As a third possibility (Section 3.3), I let $r = m$ where $m = \{m_1, ..., m_n\}$ is the vector that collects agents' minimal rights. Finally (Section 3.4), I study a setting in which reference points are given by claimants' beliefs about the awards vector that the arbitrator will implement. Formally, $r = F$ where F is a probability distribution defined over the set $V = \{P(c, E), CEA(c, E), CEL(c, E), T(c, E)\}$.

The four specifications can be classified according to different criteria. For instance, one may focus on the relation between reference points and claims. This can be direct ($r = c$), indirect ($r = m$ and $r = F$ as claims influence agents' minimal rights and beliefs), or non existing ($r = \mathbf{0}$).

Alternatively, one may consider the feasibility of the vector r and its implications on the resulting allocations and on agents' perceived gains and losses. If $r = c$, awards vectors will inevitably generate some losses at the individual level; if instead $r = \mathbf{0}$ or

$r = m$, awards vectors that lead all claimants to perceive some gains are feasible; and both gains and losses are possible when $r = F$. RDPs then imply that the welfare-maximizing allocations are such that no agent receives an amount larger than r_i when reference points cannot be matched (unless one is willing to give up *Boundedness*, see Section 4.2), whereas each agent receives at least r_i when reference points are feasible.[4]

3.1 Claims as Reference Points

Let agents' reference points be determined by their claims. Formally, let $r = c$. Claims are thus interpreted as an expression of the agents' rights, needs, demands, or aspirations (Mariotti and Villar, 2005). The fact that agents use their claims as reference points has already been explored in cooperative models of bargaining (Gupta and Livne, 1988; Thomson, 1994) and can be rationalized in different ways. For instance, agents may not be fully aware that they are involved in a bankruptcy problem and that thus rationing must necessarily take place. Alternatively, they may know that the endowment is not enough to satisfy aggregate demand, yet they may think, perhaps erroneously, that they have or deserve some sort of priority with respect to others.

UTILITARIAN WELFARE ANALYSIS

Proposition 1 ranks the Proportional rule, the Constrained Equal Awards rule, and the Constrained Equal Losses rule on the basis of the level of utilitarian welfare that they generate. The ranking holds because the rules differ on how they allocate the aggregate loss across claimants. Since agents display diminishing marginal sensitivity to losses, differences in the allocation of individual losses lead to differences in welfare.

[4] The baselines first operator proposed by Hougaard et al. (2012, 2013a, 2013b) also satisfies this property. A baseline b is an exogenously given or endogenously generated vector that serves as reference point. An operator is a mapping that associates with each rule another one. The baseline first operator maps rule R into rule R' where R' tackles the problem (c, E) in two stages. If b is feasible, R' first allocates b_i to each claimant and then allocates what remains of E according to R and the vector of adjusted claims $c' = c - b$. Thus, b is a lower bound for $R'(c, E)$. If b is unfeasible, R' first adjusts the claims vector to $c' = b$ and then uses R to solve the problem (c', E). Thus, b is an upper bound for $R'(c, E)$. In my setting, reference points do not necessarily affect award vectors, although, because of RDPs, they do affect claimants' utility and thus aggregate welfare.

PROPOSITION 1. *The ranking $W_{ut}(CEA) > W_{ut}(P) > W_{ut}(CEL)$ holds in any bankruptcy problem $(c, E) \in \mathbb{B}^N$ in which claimants have RDPs, $r = c$, and $c_i \neq c_j$ for some $i, j \in N$.*

Since the Talmud rule is a combination of the CEA and the CEL rules and the latter achieves minimal welfare (see the proof of Proposition 1 in the Appendix), its performance in terms of utilitarian welfare falls in the middle. In particular, $W_{ut}(T) \in [W_{ut}(CEL), W_{ut}(P)]$ when $\frac{C}{2} < E$, whereas $W_{ut}(T) \in [W_{ut}(P), W_{ut}(CEA)]$ when $\frac{C}{2} \geq E$. The following example illustrates all these results.

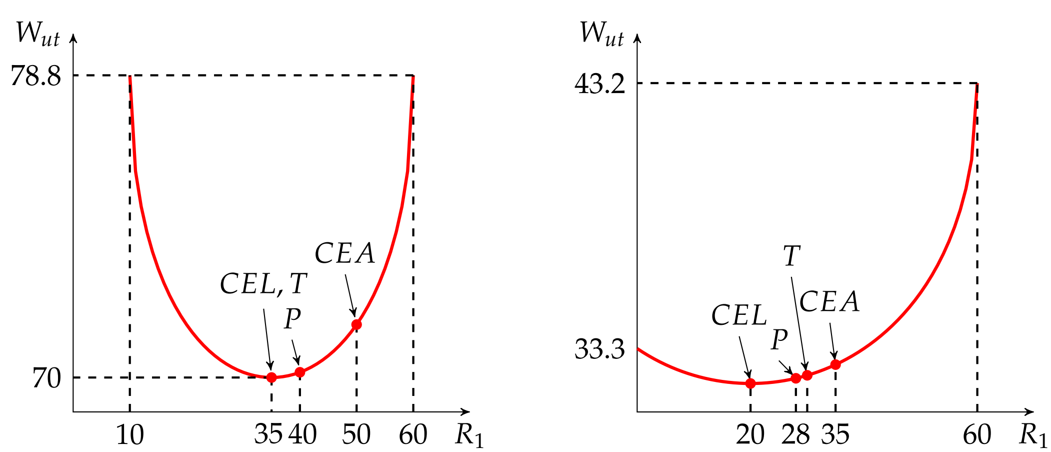
EXAMPLE 2. *Let claimant $i \in \{1, 2\}$ have utility function*

$$u(R_i(\cdot) \mid r_i = c_i) = \begin{cases} R_i(\cdot) + \sqrt{R_i(\cdot) - c_i} & \text{if } R_i(\cdot) \geq c_i \\ R_i(\cdot) - 3\sqrt{|R_i(\cdot) - c_i|} & \text{if } R_i(\cdot) < c_i \end{cases}$$

and consider the bankruptcy problems: (a) $c = (60, 90)$, $E = 100$; and (b) $c = (60, 90)$, $E = 70$.

(a) Awards vectors are $P(c, E) = (40, 60)$, $CEA(c, E) = (50, 50)$, and $CEL(c, E) = T(c, E) = (35, 65)$. Therefore, $W_{ut}(CEA) > W_{ut}(P) > W_{ut}(T) = W_{ut}(CEL)$. See Figure 1(a).

(b) Awards vectors are $P(c, E) = (28, 42)$, $CEA(c, E) = (35, 35)$, $CEL(c, E) = (20, 50)$, and $T(c, E) = (30, 40)$. Therefore, $W_{ut}(CEA) > W_{ut}(T) > W_{ut}(P) > W_{ut}(CEL)$. See Figure 1(b).



(a) W_{ut} with $c = (60, 90)$, $E = 100$.

(b) W_{ut} with $c = (60, 90)$, $E = 70$.

Figure 1: Utilitarian welfare when $r = c$.

Proposition 1 ranks standard rules according to the level of utilitarian welfare that they generate. However, the proposition remains silent on how these rules compare with the first-best solution. Clearly, an awards vector that maximizes utilitarian welfare exists since the function $W_{ut}(\cdot)$ is continuous in the closed and bounded space defined by *Boundedness* and *Balance*. Figure 1 indicates that standard rules may fail to select such a vector. The rules in fact do not properly take into account agents' diminishing sensitivity to losses. This property implies that, from a purely utilitarian point of view, it is more efficient to largely disappoint a subset of agents rather than to slightly disappoint all of them.

By elaborating on this insight, I introduce a rule that always selects an awards vector that maximizes utilitarian welfare. The rule belongs to the family of sequential priority rules (Moulin, 2000, Thomson, 2015). Let $\preceq$ denote an order on the set of claimants, i.e., a complete and transitive binary relation on N. The strict relation $\prec$ associated with $\preceq$ is defined as usual: $i \prec j$ iff $i \preceq j$ but not $j \preceq i$. The sequential priority rule associated with $\preceq$ assigns to each agent the minimum between his claim and what remains of the endowment. The rule that I propose follows the strict order $\prec_c$ that orders claimants according to their claims and starting from the lowest (ties are broken randomly). Then, $i \prec_c j$ iff $c_i < c_j$, whereas both $i \prec_c j$ and $j \prec_c i$ are equally likely if $c_i = c_j$. I label this rule the *Small Claims First* rule.

Definition 1. *Given the order $\prec_c$ defined on N, the Small Claims First (SCF) rule assigns to each agent the minimum amount between his claim and what remains of the endowment:*

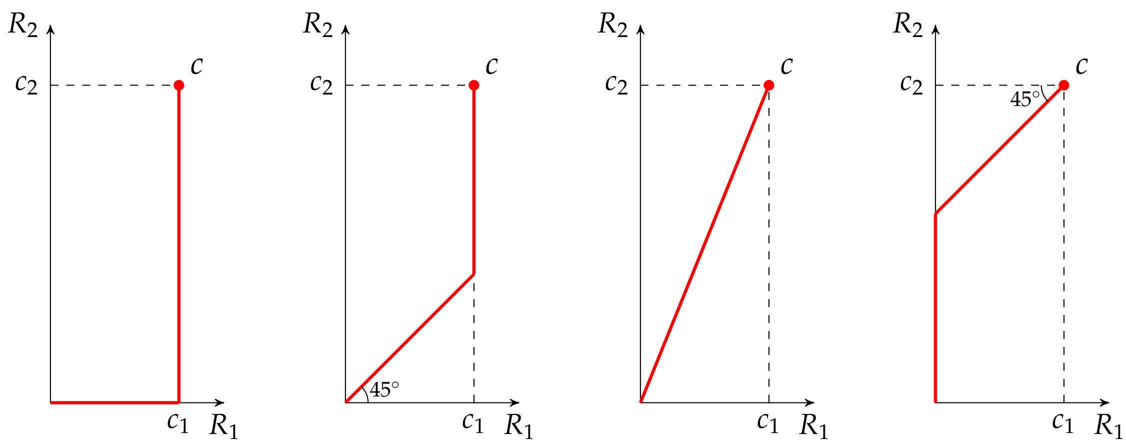
$$SCF_i(c, E) = \min\left\{c_i, \max\left\{E - \sum_{j \prec_c i} c_j, 0\right\}\right\} \quad \textit{for all } i \in N.$$

By construction, the *SCF* rule selects an awards vector that matches the claims of as many claimants as possible and disappoints the remaining claimants as much as possible.[5] Proposition 2 describes how the rule performs in terms of utilitarian welfare.

[5]For instance, if $c = (30, 50, 80)$ and $E = 100$ (cfr. Example 1), the *SCF* rule selects the awards vector $SCF(c, E) = (30, 50, 20)$ and thus attributes the entire loss to agent 3. Similarly, $SCF(c, E) = (60, 40)$ in Example 2(a), whereas $SCF(c, E) = (60, 10)$ in Example 2(b).

Proposition 2. *The SCF rule maximizes utilitarian welfare in any bankruptcy problem $(c, E) \in \mathbb{B}^N$ in which claimants have RDPs and $r = c$.*

The SCF rule thus dominates standard rules in terms of utilitarian welfare.[6] Since claimants display diminishing sensitivity to losses, the SCF rule achieves higher utilitarian welfare as it selects a more extreme allocation. This is evident if one compares the paths of awards that the various rules implement (see Figure 2). The path of awards is the locus of allocations that a rule selects as, holding fixed the claim vector c, the endowment E grows from 0 to C. Note that the order of the rules in the figure reflects their ranking in terms of utilitarian welfare.



(a) $SCF(c, E)$ (b) $CEA(c, E)$ (c) $P(c, E)$ (d) $CEL(c, E)$

Figure 2: Paths of awards with $n = 2$.

The awards vector $SCF(c, E)$ may not be the sole allocation that maximizes welfare. For instance, when there are only two claimants and $\max\{c_1, c_2\} \leq E$ then there always exist two optimal allocations (see Figure 1(a)). However, if multiple solutions exist, the SCF rule always selects a specific welfare-maximizing allocation.

Proposition 3. *Whenever there exist multiple awards vectors that maximize utilitarian welfare the SCF rule selects the one that has the lowest level of inequality.*

[6] The dominance relation holds no matter if claimants are symmetric or asymmetric. The relation is always strict with the only exception being the case in which there exist $n-1$ claimants with $c_i < E/n$ and one claimant j with $c_j > E - \sum_{i \neq j} c_i$, in which case $SCF(c, E) = CEA(c, E)$ and thus $W_{ut}(SCF) = W_{ut}(CEA)$.

The following example illustrates the result of Proposition 3.

EXAMPLE 3. *Consider the problem* (c, E) *with* $c = (10, 20, 50, 60)$ *and* $E = 100$. *Then* $SCF(c, E) = (10, 20, 50, 20)$ *with* $W_{ut}(SCF) = 1 + \mu(-40)$. *The awards vector* $R(c, E) = (10, 20, 10, 60)$ *also achieves welfare* $W_{ut}(R) = 1 + \mu(-40)$. *However,* $SCF(c, E)$ *has lower variance:* $\sigma^2(SCF(c, E)) = 225$, $\sigma^2(R(c, E)) = 425$.

In what follows, I fully characterize the SCF rule. I start by defining the notion of agent i's *Cumulative Aggregate Loss*, which measures the amount by which the sum of the claims of i and of his predecessors exceeds the endowment.

DEFINITION 2. *Given the order* $\prec_c$ *defined on* N, *the Cumulative Aggregate Loss of claimant* $i \in N$ *is given by* $\tilde{L}_i = \max\left\{\sum_{j \preceq_c i} c_j - E, 0\right\}$.

Next, I introduce two properties that rely on this definition. The first property (*Large Losers*) states that if the Cumulative Aggregate Loss of an agent is larger or equal than his claim, then the agent should get nothing.

Large Losers: For all $(c, E) \in \mathbb{B}^N$ if $\tilde{L}_i \geq c_i$ then $R_i(c, E) = 0$.

The second property (*Unique Residual Loser*) states that if the Cumulative Aggregate Loss of agent i is positive but smaller than his claim, then a unique agent (either agent i or one of his predecessors) must suffer that loss in full.

Unique Residual Loser: For all $(c, E) \in \mathbb{B}^N$, if there exists a claimant $i \in N$ such that $0 < \tilde{L}_i < c_i$ then there exists a claimant $j \preceq_c i$ such that $R_j(c, E) = c_j - \tilde{L}_i$.

These two properties suffice to characterize the set of rules that maximize utilitarian welfare.

PROPOSITION 4. *In any bankruptcy problem* $(c, E) \in \mathbb{B}^N$ *in which claimants have RDPs and* $r = c$, *a rule maximizes utilitarian welfare if and only if it satisfies Large Losers and Unique Residual Loser.*

To uniquely pin down the *SCF* rule a strengthening of *Unique Residual Loser* is needed. The property called *Unique Residual Loser Is The Last* states that if the Cumulative Aggregate Loss of agent i is positive but smaller than his claim, it is actually agent i the unique agent who suffers that loss in full.

Unique Residual Loser Is The Last: For all $(c, E) \in \mathbb{B}^N$, if there exists a claimant $i \in N$ such that $0 < \tilde{L}_i < c_i$ then $R_i(c, E) = c_i - \tilde{L}_i$.

PROPOSITION 5. *In any bankruptcy problem $(c, E) \in \mathbb{B}^N$ in which claimants have RDPs and $r = c$, the SCF rule is the only rule that satisfies Large Losers and Unique Residual Loser Is The Last.*

The following example illustrates the bite of the properties that I introduced in first identifying the rules that maximize utilitarian welfare and then characterize the *SCF* rule.

EXAMPLE 4. *Consider the problem (c, E) with $c = (10, 20, 40, 50, 60, 80)$ and $E = 100$. The vector of Cumulative Aggregate Losses is given by $\tilde{L} = (0, 0, 0, 20, 80, 160)$. Large Losers thus selects all the awards vectors such that $(\cdot, \cdot, \cdot, \cdot, 0, 0)$. Unique Residual Loser further refines the set of awards vectors to $R(c, E) = (10, 20, 40, 30, 0, 0)$, $R'(c, E) = (10, 20, 20, 50, 0, 0)$, and $R''(c, E) = (10, 0, 40, 50, 0, 0)$. These are the vectors that maximize utilitarian welfare with $W_{ut}(\cdot) = 100 + \mu(-20) + \mu(-60) + \mu(-80)$. By substituting Unique Residual Loser with Unique Residual Loser Is The Last one gets the unique vector $R(c, E) = (10, 20, 40, 30, 0, 0)$, which is the SCF solution.*

In terms of more standard properties, the *SCF* rule satisfies *Endowment Monotonicity*, *Scale Invariance*, *Path Independence*, and *Composition*. It does not satisfy *Claims Monotonicity* and *Order Preservation in Gains*.[7] More importantly, it does not satisfy *Equal Treatment of Equals*. The property says that agents with identical claims should get identical awards.

Equal Treatment of Equals: For all $(c, E) \in \mathbb{B}^N$ and all $i, j \in N$, if $c_i = c_j$ then $R_i(c, E) = R_j(c, E)$.

[7] See Thomson (2015) for a detailed description of all the properties that a rule may or may not satisfy.

The analysis thus highlights a tension between the maximization of utilitarian welfare and the equity of the resulting awards vector.[8] As such, the *SCF* rule is possibly not palatable to an arbitrator who wants to be impartial and treat symmetric claimants in the same way. There are however situations in which an arbitrator should indeed discriminate across agents, even though their claims are symmetric. For instance, there may be differences among agents that are not captured by their claims but rather stem from individual characteristics (e.g., age, gender), exogenous rights, or merits (Moulin, 2000). In these circumstances, the *SCF* rule may be appropriate to guide the choice of an arbitrator who wants to minimize the aggregate level of disappointment (i.e., the negative impact that perceived losses have on welfare).

The *SCF* rule actually satisfies a weaker form of *Equal Treatment of Equals*, which I label *Ex-Ante Equal Treatment of Equals*.

Ex-Ante Equal Treatment of Equals: For all $(c, E) \in \mathbb{B}^N$ and all $i, j \in N$, if $c_i = c_j$ then $\mathbb{E}\left(R_i\left(c, E\right)\right) = \mathbb{E}\left(R_j\left(c, E\right)\right)$.

The *SCF* rule is thus procedurally fair (Bolton et al., 2005). Ties among agents with the same claims are broken randomly in determining the priority order. Thus, the rule allocates the same *expected* award to identical claimants.[9]

EXAMPLE 5. *Consider the problem (c, E) with $c = (30, 50, 50, 80)$ and $E = 100$. Let $1 \prec_c 2 \prec_c 3 \prec_c 4$. Then, $SCF(c, E) = (30, 50, 20, 0)$ such that claimants 2 and 3 are treated differently. However, the orders $\prec_c$ such that $1 \prec_c 2 \prec_c 3 \prec_c 4$ and $\prec'_c$ such that $1 \prec'_c 3 \prec'_c 2 \prec'_c 4$ are ex-ante equally likely. Therefore, $\mathbb{E}(SCF_2(c, E)) = \mathbb{E}(SCF_3(c, E)) = 35$ so that claimants 2 and 3 are ex-ante treated equally.*

MAXMIN WELFARE ANALYSIS

If the arbitrator adopts a maxmin welfare specification (see Eq. 3) the optimal allocation is the one that maximizes the utility of the worst-off individual. With no constraints on

[8]The fact that the *SCF* rule does not satisfy *Equal Treatment of Equals* is evident when claimants are symmetric. In fact, if $c_i = c_j$ for all $i, j \in N$ then $SCF(c, E) = \left(c_i, c_i, ..., E - \sum_{j \prec_c i} c_j, 0, ..., 0\right)$.

[9]Analogously, one can also imagine a larger game in which the arbitrator chooses the specific order of priority to use by uniformly randomizing among all the orders that respect the condition $i \prec j$ iff $c_i < c_j$.

the awards vector, this allocation would then be the one that equalizes claimants' utility. *Boundedness* may however make such an allocation unfeasible. I thus define the *Minimal Utility Gap* rule as the rule that makes claimants' utility as equal as possible.

DEFINITION 3. *The Minimal Utility Gap (MUG) rule assigns to each claimant the amount $MUG_i(c, E, u)$ such as to minimize the term*

$$\max\{u(MUG_i(c, E, u) \mid r_i = c_i)\}_{i \in N} - \min\{u(MUG_i(c, E, u) \mid r_i = c_i)\}_{i \in N}.$$

Proposition 6 then immediately follows.

PROPOSITION 6. *The MUG rule maximizes maxmin welfare in any bankruptcy problem $(c, E) \in \mathbb{B}^N$ in which claimants have RDPs and $r = c$.*

Contrary to the definitions of the P, CEA, CEL, T, and SCF rules, Definition 3 provides a rather vague description of the actual awards vector that the MUG rule selects. In fact, and as made clear by the notation, the vector $MUG(c, E, u)$ depends on the profile of utility functions $u = (u_1(\cdot), ..., u_n(\cdot))$. It is thus not possible to analytically define the optimal awards vector without taking into account the specific functional form of claimants' utility functions. This makes the MUG rule perform poorly in terms of standard properties: the rule satisfies *Equal Treatment of Equals*, *Endowment Monotonicity* and *Order Preservation in Gains* but it fails *Claims Monotonicity*, *Scale Invariance*, *Path Independence*, and *Composition*. The dependence of the awards vector on claimants' utility functions also prevents a characterization of the MUG rule (a part from the trivial characterization based on the requirement that the awards vector minimizes the variance of the u vector).[10]

It is anyway possible to infer some general features of the solution $MUG(c, E, u)$. Since claimants use their claims as reference points, their utility (see Eq. 1) is given by:

$$u(MUG_i(c, E, u) \mid r_i = c_i) = MUG_i(c, E, u) + \mu(MUG_i(c, E, u) - c_i)$$

[10]Note that when *Boundedness* impedes the equalization of claimants' utility, the *MUG* rule may not be the unique rule that maximizes welfare. When this is the case, the rule however selects the awards vector that generates the less unequal profile of individual utilities.

Utility depends positively on the amount of the endowment that the claimant receives, and negatively on his claim. The optimal allocation trades off these two effects across agents. With respect to the egalitarian allocation, the MUG rule thus assigns more of the endowment to agents who have higher claims. The size of these distortions increases with the relevance that perceived losses have on claimants' overall utility. If perceived losses have limited effects (i.e., the agent's well-being is mainly determined by the actual amount of the endowment that he receives from the arbitrator) then the CEA rule, by allocating the endowment across agents as equally as possible, will outperform other standard rules.[11] If instead perceived losses have a large effect on individual utilities, distortions become sizable and can modify the ranking between the CEA rule and the other rules. The following example illustrates these results.

EXAMPLE 6. *Let claimant $i \in \{1, 2\}$ have utility function*

$$u(R_i(\cdot) \mid r_i = c_i) = \begin{cases} R_i(\cdot) + \sqrt{R_i(\cdot) - c_i} & \text{if } R_i(\cdot) \geq c_i \\ R_i(\cdot) - 3\sqrt{|R_i(\cdot) - c_i|} & \text{if } R_i(\cdot) < c_i \end{cases}$$

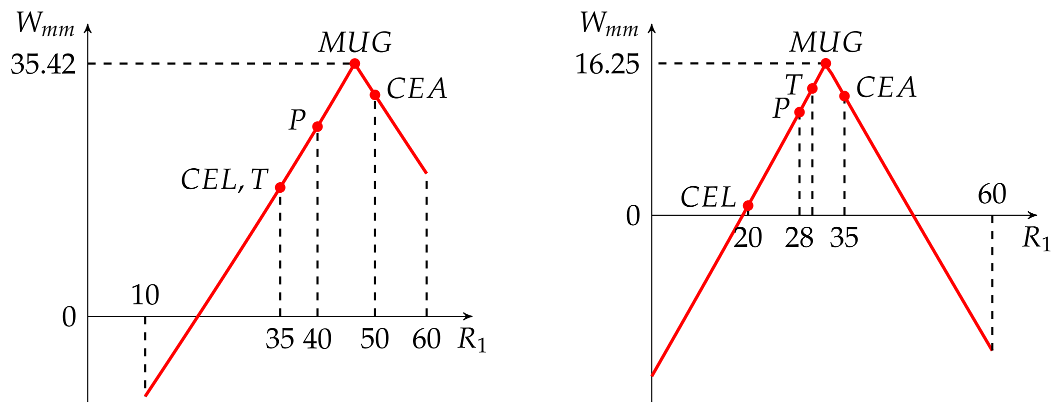
and consider the bankruptcy problems: (a) $c = (60, 90)$, $E = 100$ and (b) $c = (60, 90)$, $E = 70$.

(a) Awards vectors are $P(c, E) = (40, 60)$, $CEA(c, E) = (50, 50)$, $CEL(c, E) = T(c, E) = (35, 65)$, and $MUG(c, E, u) \approx (46.5, 53.5)$. Thus, $W_{mm}(MUG) > W_{mm}(CEA) > W_{mm}(P) > W_{mm}(CEL) = W_{mm}(T)$. See Figure 3(a).

(b) Awards vectors are $P(c, E) = (28, 42)$, $CEA(c, E) = (35, 35)$, $CEL(c, E) = (20, 50)$, $T(c, E) = (30, 40)$, and $MUG(c, E, u) \approx (32.1, 37.9)$. Therefore, $W_{mm}(MUG) > W_{mm}(T) > W_{mm}(CEA) > W_{mm}(P) > W_{mm}(CEL)$.[12] See Figure 3(b).

[11] With respect to the first-best solution (i.e., $MUG(c, E)$), the CEA rule allocates less (more) of the endowment to agents that have higher (lower) claims.

[12] As usual, $T(c, E)$ is bounded by $CEA(c, E)$ and $CEL(c, E)$. However, because of the shape of function W_{mm}, in problem (b) the T rule achieves higher welfare than the CEA and the CEL rules.



(a) W_{mm} with $c = (60, 90)$, $E = 100$.

(b) W_{mm} with $c = (60, 90)$, $E = 70$.

Figure 3: Maxmin welfare when $r = c$.

3.2 Zero Awards as Reference Point

Let agents have null reference points ($r = \mathbf{0}$). The setting is appropriate to describe all those situations in which agents do have claims on the endowment E but still consider them to be worthless, perhaps because they think that there is nothing to share (i.e., $E = 0$). Examples include the case of creditors who expect the bankrupted firm not to have any asset left, or heirs who are not aware of the deceased's net worth. Claimants are then pleasantly surprised whenever they receive an award $R_i(c, E) > 0$. The relevant part of the $\mu(\cdot)$ function is thus the domain of gains: each agent experiences a perceived gain of size $g_i = R_i(c, E) - 0 \geq 0$ and this increases utility by the amount $\mu(R_i(c, E)) \geq 0$.

UTILITARIAN AND MAXMIN WELFARE ANALYSIS

Since agents are now perfectly symmetric (they all have the same reference point) and RDPs postulate diminishing marginal sensitivity to gains, the rules that select the most egalitarian awards vectors achieve higher levels of welfare. Proposition 7 ranks the P, CEA, and CEL rules. The CEA rule not only dominates the other rules, it actually implements the first-best solution under both welfare specifications.

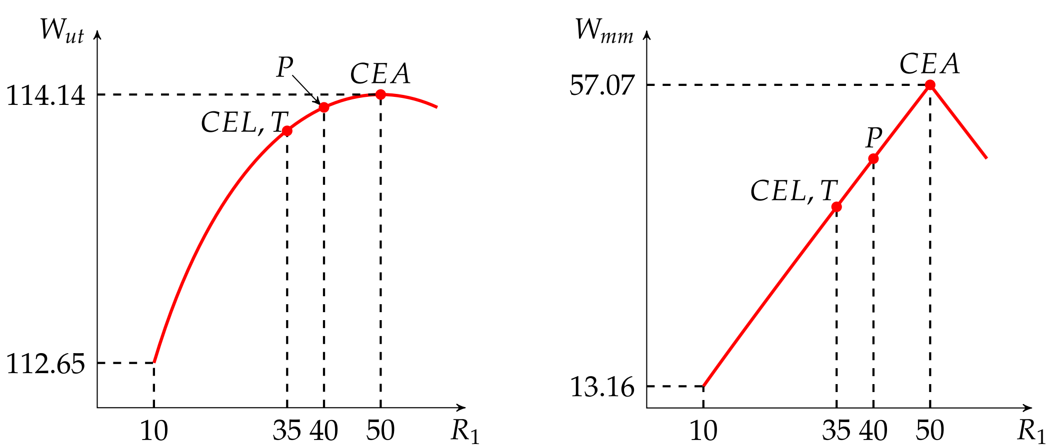
Proposition 7. *The ranking $W_w(CEA) > W_w(P) > W_w(CEL)$ with $w \in \{ut, mm\}$ holds in any bankruptcy problem $(c, E) \in \mathbb{B}^N$ in which claimants have RDPs, $r = \mathbf{0}$, and $c_i \neq c_j$ for* *some $i, j \in N$. In particular, the CEA rule achieves maximal (utilitarian and maxmin) welfare.*

The performance of the Talmud rule is such that $W_w(T) \in [W_w(CEL), W_w(CEA)]$ for any $w \in \{ut, mm\}$, with $W_w(T) < W_w(P)$ if $\frac{C}{2} < E$ and $W_w(T) \geq W_w(P)$ if $\frac{C}{2} \geq E$. The following example illustrates the results of Proposition 7.[13]

EXAMPLE 7. *Let claimant $i \in \{1, 2\}$ have utility function*

$$u(R_i(\cdot) \mid r_i = 0) = \begin{cases} R_i(\cdot) + \sqrt{R_i(\cdot)} & \text{if } R_i(\cdot) \geq 0 \\ R_i(\cdot) - 3\sqrt{|R_i(\cdot)|} & \text{if } R_i(\cdot) < 0 \end{cases}$$

and consider the bankruptcy problem (c, E) with $c = (60, 90)$ and $E = 100$.
Awards vectors are $P(c, E) = (40, 60)$, $CEA(c, E) = (50, 50)$, and $CEL(c, E) = T(c, E) = (35, 65)$. Therefore, $W_w(CEA) > W_w(P) > W_w(T) = W_w(CEL)$ for both $w \in \{ut, mm\}$. See Figures 4(a) and 4(b).



(a) W_{ut} with $c = (60, 90)$, $E = 100$.

(b) W_{mm} with $c = (60, 90)$, $E = 100$.

Figure 4: Utilitarian and maxmin welfare when $r = \mathbf{0}$.

[13] With respect to Examples 2 and 6, Example 7 only considers the problem (c, E) with $c = (60, 90)$ and $E = 100$. The graphical analysis of the problem (c, E) with $c = (60, 90)$ and $E = 70$ leads to similar results and it is therefore omitted.

3.3 Minimal Rights as Reference Points

Consider now the case in which agents' reference points are determined by their minimal rights. A claimant's minimal right is given by what remains of the endowment (if anything) after all other agents get their claims fully honored. The minimal right of agent i can thus be interpreted as the minimum amount that i can reasonably expect to get (Thomson and Yeh, 2008). Formally, let $r = m$ where $m = (m_1, ..., m_n)$ and $m_i = \max\left\{E - \sum_{j \neq i} c_j, 0\right\}$ for any $i \in N$. Clearly, if $m = 0$ the problem is analogous to the one analyzed in Section 3.2 and thus Proposition 7 applies. I thus focus on the situation in which the vector m is such that $m_i > 0$ for some $i \in N$.

UTILITARIAN WELFARE ANALYSIS

It is always possible for the arbitrator to implement an allocation that matches (and possibly trespasses) the minimal rights of *all* the claimants.[14] Because of the properties of the $\mu(\cdot)$ function (losses loom larger than gains), such an allocation will be welfare superior to any allocation in which $R_i(c, E) < m_i$ for some $i \in N$.

Every claimant $i \in N$ will thus experience a perceived gain of size $g_i = R_i(c, E) - m_i \geq 0$. The diminishing marginal sensitivity to gains of $\mu(\cdot)$ then implies that the optimal utilitarian allocation is the one that minimizes the variance of the vector $g = (R(c, E) - m)$. This is what the *Constrained Equal Gains* rule accomplishes. Proposition 8 then immediately follows.

Definition 4. *Given the order $\prec_c$ defined on N, the Constrained Equal Gains (CEG) rule assigns to each agent $i \in N$ the amount:*

$$CEG_i(c, E) = \min\left\{c_i, m_i + \frac{E - \sum_{j \succeq_c i} m_j - \sum_{j \prec_c i} CEG_j}{n - i + 1}\right\}.$$[15]

[14] To see this, assume first that all claimants have strictly positive minimal rights: $m_i > 0$ for all $i \in N$. Then, $\sum_i m_i = nE - (n-1)C$ such that $E - \sum_i m_i = (n-1)(C - E) \geq 0$. Therefore, $E \geq \sum_i m_i$ (which obviously also holds if $m_i = 0$ for some $i \in N$) and an awards vector $R(c, E) \geq m$ is thus feasible.

[15] Although the functional form of the CEG rule is rather complex, its intuition is clear. Consider first the simpler rule CEG^* which assigns to each claimant the amount $CEG^*_i(c, E) = m_i + \frac{E - \sum_j m_j}{n}$. The rule maximizes utilitarian welfare whenever $CEG^*_i(c, E) \leq c_i$ for all $i \in N$ since $g(CEG^*) = \left(\frac{E - \sum_j m_j}{n}, ..., \frac{E - \sum_j m_j}{n}\right)$ and thus $\sigma^2(g(CEG^*)) = 0$. The more general formulation of the CEG rule takes into account the fact

PROPOSITION 8. *The CEG rule maximizes utilitarian welfare in any bankruptcy problem* $(c, E) \in \mathbb{B}^N$ *in which claimants have RDPs and* $r = m \geq 0$.

Example 8 illustrates the result. It also shows that no clear ranking of the P, CEA, CEL, and T rules exists.

EXAMPLE 8. *Let claimant* $i \in \{1, 2\}$ *have utility function*

$$u(R_i(\cdot) \mid r_i = m_i) = \begin{cases} R_i(\cdot) + \sqrt{R_i(\cdot) - m_i} & \text{if } R_i(\cdot) \geq m_i \\ R_i(\cdot) - 3\sqrt{|R_i(\cdot) - m_i|} & \text{if } R_i(\cdot) < m_i \end{cases}$$

and consider the bankruptcy problems: (a) $c = (60, 90)$, $E = 100$; *and (b)* $c = (60, 90)$, $E = 70$.

(a) The vector of minimal rights is $m = (10, 40)$. *Awards vectors are* $P(c, E) = (40, 60)$, $CEA(c, E) = (50, 50)$, $CEL(c, E) = (35, 65)$, $T(c, E) = (35, 65)$, *and* $CEG(c, E) = (35, 65)$. *Therefore,* $W_{ut}(CEG) = W_{ut}(CEL) = W_{ut}(T) > W_{ut}(P) > W_{ut}(CEA)$. *See Figure 5(a).*

(b) The vector of minimal rights is $m = (0, 10)$. *Awards vectors are* $P(c, E) = (28, 4)$, $CEA(c, E) = (35, 35)$, $CEL(c, E) = (20, 50)$, $T(c, E) = (30, 40)$ *and* $CEG(c, E) = (30, 40)$. *Therefore,* $W_{ut}(CEG) = W_{ut}(T) > W_{ut}(P) > W_{ut}(CEA) > W_{ut}(CEL)$.[16] *See Figure 5(b).*

In terms of standard properties, the CEG rule satisfies *Equal Treatment of Equals*, *Endowment Monotonicity*, *Claims Monotonicity*, *Order Preservation in Gains*, *Scale Invariance*, and *Path Independence*. It fails *Composition*. I do not provide a characterization of the rule as it would be too ad hoc.

MAXMIN WELFARE ANALYSIS

If the arbitrator follows maxmin welfare, the *Minimal Utility Gap* (MUG) rule (see Definition 3) is the optimal rule.

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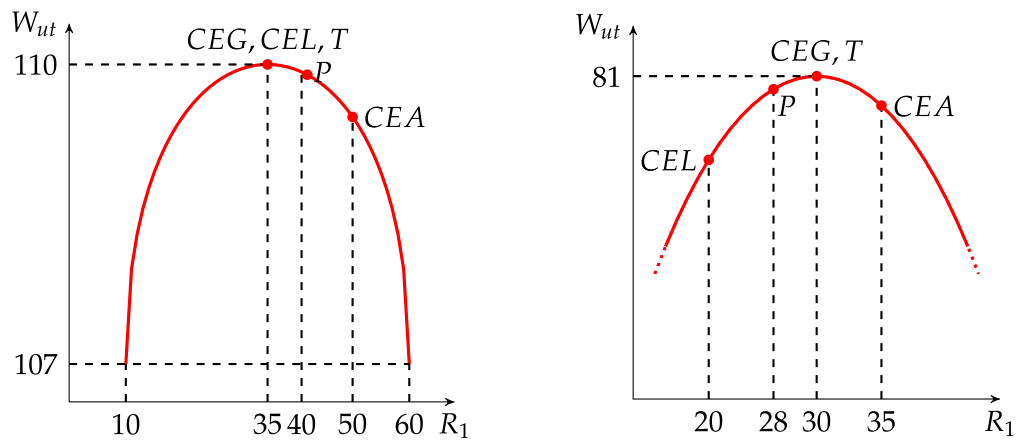
that awards vector must satisfy *Boundedness*. Thus, if $CEG_i^*(c, E) > c_i$ for some $i \in N$, the amount $CEG_i^*(c, E) - c_i > 0$ must be redistributed among the other claimants in a welfare-maximizing way.

[16] The fact that $T(c, E) = CEG(c, E)$ in both problems is a peculiarity of the two claimants case and does not hold in general. For instance, if $c = (10, 40, 70)$ and $E = 100$ then $T(c, E) = (5, 32.5, 62.5)$ and $CEG(c, E) = (10, 30, 60)$ with $W_{ut}(CEG) > W_{ut}(T)$.



(a) W_{ut} with $c = (60, 90)$, $E = 100$. (b) W_{ut} with $c = (60, 90)$, $E = 70$.

Figure 5: Utilitarian welfare when $r = m \geq 0$.

Proposition 9. *The MUG rule maximizes maxmin welfare in any problem $(c, E) \in \mathbb{B}^N$ in which claimants have RDPs and $r = m$.*

With respect to the egalitarian allocation, the MUG rule allocates more of the endowment to claimants who have higher minimal rights. The intuition is that these agents will experience lower perceived gains and must thus be compensated with a relatively higher allocation of the endowment. As it was the case in Section 3.1, if the relevance of perceived gains is limited then the awards vector $MUG(c, E, u)$ will be close to the egalitarian allocation. Thus, the CEA rule will outperform other standard rules. Different rankings can instead emerge when the impact of perceived gains on claimants' total utility is sizable.[17]

3.4 Expected Awards as Reference Points

As a last specification, I let claimants' reference points be determined by their expectations about what the arbitrator will do. Say that claimants hold (common) beliefs about final outcomes, i.e., about the awards vector (analogously, the rule) that the arbitrator will select. These beliefs are described by the probability distribution F defined over the set of vectors $V = \{P(c, E), CEA(c, E), CEL(c, E), T(c, E)\}$ and with density f.

[17] I avoid replicating the illustrative example and the figures as the situation is similar to the one described in Section 3.1 for the case $r = c$ (see Example 6 and Figure 3).

Following Kőszegi and Rabin (2007), I let claimants' reference points coincide with their beliefs. Formally, $r = F$. A claimant's evaluation of a *given* award $R_i(c, E) \in V$ is thus given by:

$$U(R_i(c,E) \mid r = F) = \int u(R_i(c,E) \mid v_i) \quad dF(v) \tag{4}$$

where $v \in V$ and $u(R_i(c,E) \mid v_i)$ is as defined in (1). The formulation thus considers how $R_i(c,E)$ compares with *all* the possible alternatives in V.[18]

I first consider the case of F being a degenerate probability distribution, so that $f(v) = 1$ for some $v \in V$. Claimants thus expect the arbitrator to implement a specific awards vector, perhaps because the latter publicly announced the rule that he intends to use or built a reputation for always using a certain rule. I then let F be a non-degenerate distribution, so that claimants are indeed uncertain about what the arbitrator will do.[19]

UTILITARIAN WELFARE ANALYSIS

If agents expect the arbitrator to implement the award vector $R(c,E)$, then it is indeed R the rule that maximizes utilitarian welfare. The intuition is that if claimants have correct expectations about what the arbitrator will do they will experience no perceived gains or losses. Because of the properties of the $\mu(\cdot)$ function, the award vector $R(c,E)$ thus dominates any alternative allocation that does not match claimants' expectations.

Proposition 10. *Generic rule $R \in \{P, CEA, CEL, T\}$ maximizes utilitarian welfare in any bankruptcy problem $(c,E) \in \mathbb{B}^N$ in which claimants have RDPs, $r = F$, and $f(R(c,E)) = 1$.*

[18] A claimant's expected utility instead evaluates *all* possible outcomes in light of *all* possible reference points (see again Kőszegi and Rabin, 2007). Since both random variables are distributed according to F, expected utility is given by:

$$U(F \mid r = F) = \int \int u(R_i(c,E) \mid v_i) \quad dF(v) \quad dF(R_i(c,E)). \tag{5}$$

However, as aggregate welfare is determined by the actual utility that claimants experience, the arbitrator uses agents' ex-post evaluations (as defined in (4)) as inputs of the social welfare functions.

[19] Claimants thus face uncertainty about the arbitrator's type. Habis and Herings (2013) study bankruptcy problems that are instead stochastic in the value of the endowment and in the value of agents' claims. Habis and Herings (2013) associate to any stochastic bankruptcy problem a state-dependent transferable utility game and then test the stability of standard rules to uncertainty. Interestingly, they also find that the CEA rule is "superior" to other rules, as in their setting the CEA rule emerges as the only stable rule.

The statement of Proposition 10 conveys two interesting implications. First, it is welfare improving for the arbitrator to communicate (or build a reputation for) which rule he will adopt. Second, once that the arbitrator announces his policy R he should not deviate from it. In particular, he should resist the lobbying that may come from those claimants who would be better off under an alternative rule $R' \neq R$. What these individuals would gain under rule R' gets more than compensated by the perceived losses that other claimants would suffer.

If the distribution F is non-degenerate results are less clear-cut. By continuity, generic rule $R \in \{P, CEA, CEL, T\}$ remains the optimal rule when claimants are reasonably confident that the arbitrator will use it (i.e., $f(R(c, E)) = 1$ is high enough). When instead agents' beliefs are more diffuse, the rule that maximizes welfare varies depending on the parameters of the problem. To see this, note that rules R and R' generate utilitarian welfare:

$$W_{ut}(R) = E + \sum_{i \in N} \left(\int \mu(R_i(c, E) - v) \quad dF(v) \right) \tag{6}$$

$$W_{ut}(R') = E + \sum_{i \in N} \left(\int \mu(R'_i(c, E) - v) \quad dF(v) \right) \tag{7}$$

where $v \in V$ and $\mu(\cdot) = 0$ for $v = R_i(c, E)$ and $v = R'_i(c, E)$ respectively. Because of the properties of the gain-loss function the last term in both equations is strictly negative. However, a univocal ranking of $W_{ut}(R)$ and $W_{ut}(R')$ does not exist as it is affected by the specific numerical values of the awards vectors in V (and thus by agents' original claims) and by the distribution of beliefs.

MAXMIN WELFARE ANALYSIS

Contrary to the previous section, when F is such that $f(R(c, E)) = 1$ for some $R(c, E) \in V$, a deviation by the arbitrator from the announced policy R may increase welfare when this takes the maxmin specification and the deviation improves the well-being of the worst-off individual. As such, rule R does not necessarily maximize welfare when claimants expect the arbitrator to use it.

Standard rules however satisfy *Order Preservation in Gains*. The order of awards thus reflects the order of claims such that the worst-off individual is the agent with the lowest claim. By construction, the CEA rule is the most generous one towards the claimant with the lowest claim as it allocates him the award $CEA_i(c, E) = \min\{c_i, E/n\}$. Then, if agents expect the arbitrator to implement the CEA allocation, the CEA rule indeed maximizes welfare. Any deviation to a different rule decreases the utility of the worst-off individual: not only the alternative rule assigns to the agent less of the endowment, it also inflicts him a loss as the actual amount that the agent gets falls short of his expectations.

PROPOSITION 11. *The CEA rule maximizes maxmin welfare in any bankruptcy problem* $(c, E) \in \mathbb{B}^N$ *in which claimants have RDPs,* $r = F$*, and* $f(CEA(c, E)) = 1$.

The result of Proposition 11 partly extends to the setting in which F is non-degenerate. As said, the CEA rule is the rule that allocates the largest amount to the agent who gets the least. Moreover, the actual realization of $CEA(c, E)$ generates some additional pleasant feelings of perceived gains as the agent compares $CEA_i(c, E)$ with all other (less favorable) awards he could have got. And the more unlikely the outcome $CEA(c, E)$ was, the larger are these positive effects. As such, the CEA rule maximizes welfare for a wide range of parameters. It may however happen that the perceived losses that claimant $j \neq i$ experiences because of the realization of the vector $CEA(c, E)$ makes j become the worst-off individual. In these circumstances, it is no more necessarily the case that the CEA rule is the optimal rule.

4. Extensions

In this section I discuss some additional topics of interest and extensions of the baseline model.

4.1 Duality

How do standard duality results get affected when claimants have RDPs? To answer this question I define as a *Bankruptcy Problem with Reference Points* a triplet (c, E, r). With respect to the notation (c, E) that I used so far, the new notation highlights the role that agents' reference points may play in determining the awards vectors that different rules select.[20] I can then immediately define the notions of dual problems and dual rules.

Definition 5. *The dual of problem (c, E, r) is given by problem $(c, L, c - r)$ and two rules R and R^* are said to be dual if $R(c, E, r) = c - R^*(c, L, c - r)$.*

The vector $(c - r)$ thus collects agents' *Dual Reference Points*. The interpretation is that if in problem (c, E, r) a claimant expects to get r_i units of the endowment E, then in the dual problem $(c, L, c - r)$ the agent expects to bear $(c_i - r_i)$ units of the loss L.

Example 9. *Consider the bankruptcy problem $(c, E) = ((60, 90), 100)$ and let $(c, E, r) = ((60, 90), 100, (50, 50))$ be the associated bankruptcy problem with reference points.[21] The dual problem of (c, E, r) is thus given by $(c, L, c - r) = ((60, 90), 50, (10, 40))$.*

The definition of dual rules embeds standard duality results. Agents' reference points in fact do not affect the awards vectors that classical rules select. Thus, the Proportional rule and the Talmud rule are still self-dual, whereas the Constrained Equal Awards rule and the Constrained Equal Losses rule remain dual of each other (see Section 2.1).

The analysis however showed that reference points influence the awards vectors that some other rules select. When this is the case, Definition 5 leads to novel duality results. For instance, when claimants use their claims as reference points (see Section 3.1), the dual of the *Small Claims First* rule is called the *Large Claims First* rule and is defined as follows.

Definition 6. *Given the order $\prec_c$ defined on N, the Large Claims First (LCF) rule assigns to each agent and starting from the last the minimum amount between his claim and what remains of the endowment. Formally:*

$$LCF_i(c, E) = \min\left\{c_i, \max\left\{\sum_{j \preceq_c i} c_j - L, 0\right\}\right\} \quad \textit{for all } i \in N.$$

[20] The notation resembles the notation (c, E, b) as introduced by Hougaard et al. (2012, 2013a, 2013b) in the context of bankruptcy problems with baselines.

[21] For instance, claimants may have set $r = F$ where F is such that $f(CEA(c, E)) = 1$ (see Section 3.4).

Example 10 shows that the *SCF* rule and the *LCF* rule are dual of each other, i.e., $SCF(c, E, r) = c - LCF(c, L, c - r)$ and $LCF(c, E, r) = c - SCF(c, L, c - r)$.

EXAMPLE 10. *Consider the bankruptcy problem* $(c, E, r) = ((30, 50, 80), 100, (30, 50, 80))$. *It then follows that* $SCF(c, E, r) = (30, 50, 20)$ *and* $LCF(c, E, r) = (0, 20, 80)$.
In the dual problem $(c, L, c - r) = ((30, 50, 80), 60, (0, 0, 0))$ *the two rules instead lead to the awards vectors* $SCF(c, L, c - r) = (30, 30, 0)$ *and* $LCF(c, L, c - r) = (0, 0, 60)$.

Similarly to the *SCF* rule, the *LCF* rule fails *Equal Treatment of Equals* but satisfies its weaker version, *Ex-Ante Equal Treatment of Equals*. Notice however that duality has no implications on how a rule performs in terms of welfare. The *SCF* rule maximizes utilitarian welfare when $r = c$ (see Proposition 2). Still, it is not true that the *LCF* rule achieves minimal welfare.[22]

4.2 No Boundedness

The literature on bankruptcy problems defines as possible awards vectors all the allocations that satisfy *Boundedness* and *Balance* (see for example Moulin, 2000, Herrero and Villar, 2001, Thomson, 2003 and 2015, or Hougaard et al., 2013). The principles that these two properties embed are thus so fundamental that allocations that do not fulfill them are not even considered eligible as possible solutions to the problem.

In line with this view, I so far discussed the issue of welfare maximization only within the set of allocations that satisfy these properties. However, while it is obvious that with monotonically increasing preferences *Balance* is a necessary condition for welfare maximization, my analysis suggests that *Boundedness* may sometimes act as a constraint toward this goal. The (utilitarian or maxmin) social welfare function may in fact achieve a global maximum outside of the domain defined by this condition. Example 11 illustrates the situation.[23]

[22]For instance, $LCF(c, E, r) = (10, 90)$ in Figure 1(a), whereas $LCF(c, E, r) = (0, 70)$ in Figure 1(b).

[23]In the example claimants use their claims as reference points and the arbitrator cares about utilitarian welfare. In particular, problem (a) in Example 11 is analogous to problem (a) in Example 2. Similar examples can be constructed for maxmin welfare and for other specifications of claimants' reference points.

EXAMPLE 11. *Let claimant $i \in \{1,2\}$ have utility function*

$$u(R_i(\cdot) \mid r_i = c_i) = \begin{cases} R_i(\cdot) + k\sqrt{R_i(\cdot) - c_i} & \text{if } R_i(\cdot) \geq c_i \\ R_i(\cdot) - 3\sqrt{|R_i(\cdot) - c_i|} & \text{if } R_i(\cdot) < c_i \end{cases}$$

and consider the problem $c = (60,90)$ and $E = 100$ in two different settings: (a) $k = 1$; and (b) $k = 2$. In (a) the allocations that maximize utilitarian welfare are $R^* = (4,96)$ and $R^{**} = (66,34)$ – see Figure 6(a) –, whereas in (b) the unique maximizing allocation is $R^* = (100,0)$ – see Figure 6(b).

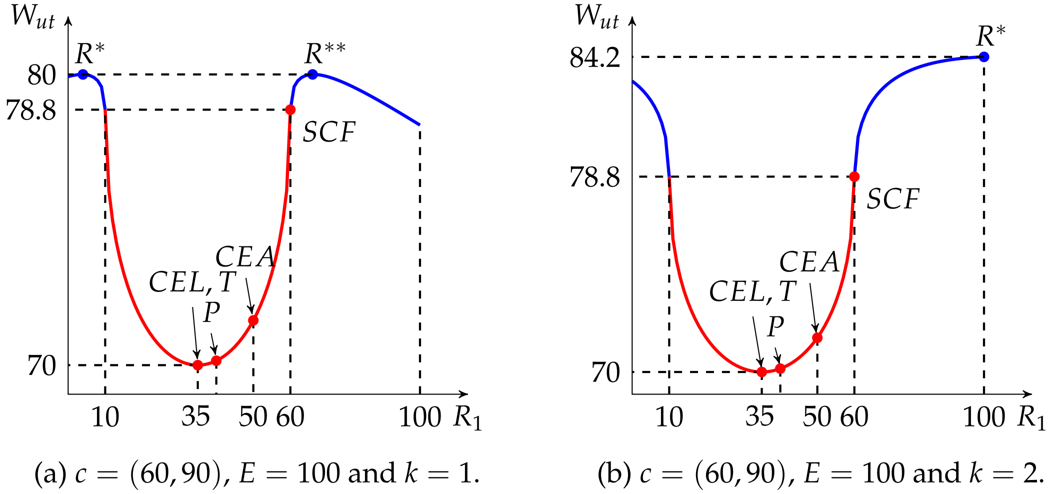


Figure 6: Utilitarian welfare when $r = c$ and *Boundedness* does not necessarily hold.

Figure 6(a) shows that welfare gets maximized by two allocations that fail *Boundedness*, as both of them assign to one of the claimant more than his claim. Figure 6(b) shows that in some circumstances welfare is maximal when the arbitrator allocates the entire endowment to a single claimant, in this case claimant 1. All these allocations however assign to some of the claimants less than their minimal rights, $m = (10,40)$. As such, it would be hard for the arbitrator to actually implement them as some agents would perceive these solutions as extremely unfair. The compliance to allocate to each claimant (at least) his minimal rights thus provides a welfare rationale for *Boundedness*.

4.3 Heterogeneous Gain-Loss Functions

The utility function defined in Section 2.2 postulates that agents have a symmetric gain-loss function: $\mu_i(\cdot) = \mu(\cdot)$ for any $i \in N$. Here I study the implications of assuming heterogeneous gain-loss functions. I thus adopt the following utility specification:

$$u_i(R_i(c, E) \mid r_i) = R_i(c, E) + \mu_i(R_i(c, E) - r_i) \tag{8}$$

where $\mu_i(\cdot)$ is now idiosyncratic to agent $i \in N$ but still obeys the general properties defined in Section 2.2.

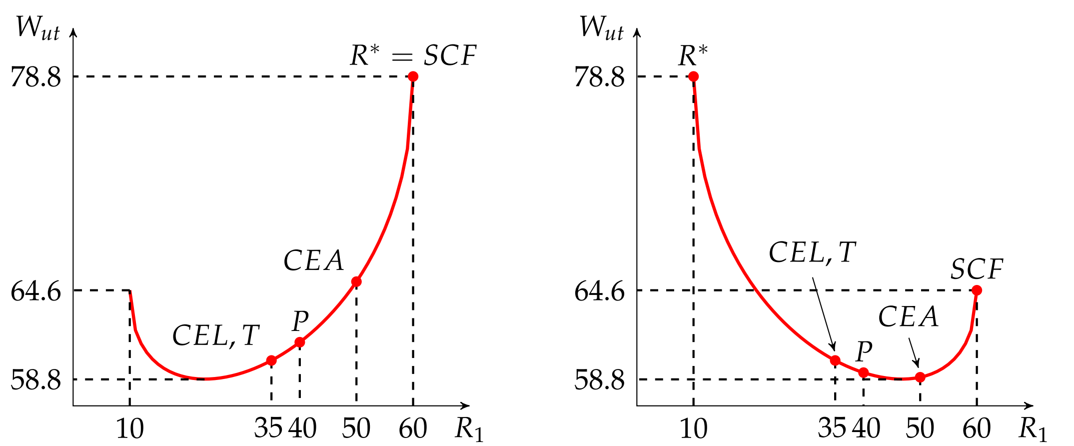
How different rules perform in terms of aggregate welfare continues to be driven by agents' diminishing sensitivity to gains and losses. However, the strength of these effects now differs across claimants and this can affect some of the results. The implications on maxmin welfare are minimal. By construction, the *MUG* rule (see Definition 3) maximizes the well-being of the worst-off individual and thus achieves maximal welfare even when claimants have different gain-loss functions. The only difference is that when claimants have a null reference point (i.e., $r = \mathbf{0}$) the *CEA* rule (which in Section 3.2 coincided with the *MUG* rule) does not necessarily achieve the first-best solution and is thus dominated by the *MUG* rule.

The implications on utilitarian welfare are more substantial. Consider for instance the case $r = c$ so that award vectors fall in the domain of losses. Diminishing sensitivity still leads to a strictly convex social welfare function. This in turn implies that award vectors that heavily punish a subset of claimants achieve higher welfare with respect to more egalitarian vectors. Heterogeneity in claimants' gain-loss functions may affect the identity of the agents that should bear the loss. In the main analysis this set simply consisted of those with the largest reference points. With heterogeneous gain-loss functions, it instead comprises those who have the "best" combination of reference point and gain-loss function, i.e., a combination that allows the arbitrator to attribute them large perceived losses without their disappointment impacting on aggregate welfare that much. Depending on how these two effects combine, this new channel may either reinforce or overturn the ranking of standard rules and the optimality of the *SCF* rule.

Example 12. *Let claimant $i \in \{1,2\}$ have utility function*

$$u_i(R_i(\cdot) \mid r_i = c_i) = \begin{cases} R_i(\cdot) + \sqrt{R_i(\cdot) - c_i} & \text{if } R_i(\cdot) \geq c_i \\ R_i(\cdot) - \gamma_i \sqrt{|R_i(\cdot) - c_i|} & \text{if } R_i(\cdot) < c_i \end{cases}$$

and consider the problem $c = (60, 90)$ and $E = 100$ in two different settings: (a) $\gamma_1 = 5$ and $\gamma_2 = 3$; and (b) $\gamma_1 = 3$ and $\gamma_2 = 5$. In (a) the awards vector that maximize utilitarian welfare is $R^* = SCF = (60, 40)$ – see Figure 7(a) –, whereas in (b) it is $R^* = (10, 90)$ – see Figure 7(b).



(a) $c = (60, 90)$, $E = 100$, $\gamma_1 = 5$, $\gamma_2 = 3$. (b) $c = (60, 90)$, $E = 100$, $\gamma_1 = 3$, $\gamma_2 = 5$.

Figure 7: Utilitarian welfare when $r = c$ and claimants have different gain-loss functions.

The logic is similar when awards vectors fall in the domain of gains (the cases $r = \mathbf{0}$ and $r = m$). The social welfare function is strictly concave and diminishing sensitivity makes egalitarian allocations achieve higher levels of welfare. The optimal awards vector however is no longer the vector that makes claimants' awards (the case $r = \mathbf{0}$, see Section 3.2) or perceived gains (the case $r = m$, see Section 3.3) as equal as possible, but rather the vector that makes their marginal benefits as equal as possible.

More in general, albeit the assumption of heterogeneous gain-loss functions seems certainly appropriate in many contexts, it makes the goal of welfare maximization more demanding from an informational point of view. In fact, not only the arbitrator should know claimants' reference points, he should also know their idiosyncratic gain-loss functions. As such, the analysis that postulates a symmetric $\mu(\cdot)$ function can be justified through a “behind the veil of ignorance” argument or interpreted as a viable heuristics that an arbitrator may use.

5. Conclusions

I studied bankruptcy problems when claimants have reference-dependent preferences. The setting is natural and leads to important welfare implications that I explored under different specifications of claimants’ reference points and different measures of welfare.

Within the class of standard rules that satisfy *Equal Treatment of Equals*, the Constrained Equal Awards rule often outperforms other rules. Focusing on utilitarian welfare, this happens when agents’ reference points coincide with their claims or with the zero awards vector. Focusing on maxmin welfare, this happens when agents’ reference points coincide with the zero awards vector or with their beliefs about the awards vector that the arbitrator will implement. It also verifies when perceived losses have second order effects and claimants use as reference points their claims or their minimal rights.

It is, however, often the case that none of the standard rules maximize welfare. I thus introduced some new rules (the Small Claims First, Minimal Utility Gap, and Constrained Equal Gains rules) that implement the first-best allocations, discussed their properties, and highlighted a tension between the goal of welfare maximization and the equity of the resulting awards vectors. The findings shed further light on the welfare implications of reference-dependent preferences that can be relevant also beyond the realm of bankruptcy problems.

6. Appendix

Proof of Proposition 1

Let $(c, E) \in \mathbb{B}^N$ and $r = c$. Generic rule R selects the awards vector $R(c, E)$ and generates utilitarian welfare $W_{ut}(R) = E + \sum_{i \in N} \mu(-l_i(R))$, where $l_i(R) = c_i - R_i(c, E) \geq 0$ is claimant i's loss. Aggregate loss is $L = \sum_{i \in N} l_i(R) = C - E$ and mean loss is $\bar{l}(R) = \frac{L}{n}$ for any R. Therefore, rules only differ in how they allocate individual losses, holding fixed aggregate loss and mean loss. Let R and R' be two rules, $l(R)$ and $l\left(R'\right)$ be the vectors of individual losses, and $\sigma^2(l(R))$ and $\sigma^2\left(l\left(R'\right)\right)$ be the variance of the elements of $l(R)$ and $l\left(R'\right)$. Without loss of generality, let $\sigma^2(l(R)) > \sigma^2\left(l\left(R'\right)\right)$. Then $l(R)$ is a mean-preserving spread of $l\left(R'\right)$ given that $\sum_{i \in N} l_i(R) = \sum_{i \in N} l_i\left(R'\right) = L$ and $\bar{l}(R) = \bar{l}\left(R'\right) = \frac{L}{n}$. Because of the strict convexity of the $\mu(\cdot)$ function in the domain of losses, it follows that $\sum_{i \in N} \mu\left(-l_i\left(R'\right)\right) < \sum_{i \in N} \mu\left(-l_i(R)\right) < 0$ and thus $W_{ut}(R) > W_{ut}\left(R'\right)$. It is thus sufficient to show that $\sigma^2(l(R)) > \sigma^2\left(l\left(R'\right)\right)$ to prove that $W_{ut}(R) > W_{ut}\left(R'\right)$.

Consider now the P, CEA, and CEL rules. By construction, the CEL rule allocates L as equally as possible. Given that $CEL(c, E) \neq R(c, E)$ for $R \in \{P, CEA\}$ whenever $c_i \neq c_j$ for some $i, j \in N$, it follows that $l(CEL) \neq l(R)$. It then must be the case that $\sigma^2(l(R)) > \sigma^2(l(CEL))$ for any $R \in \{P, CEA\}$. Therefore, $\min\{W_{ut}(P), W_{ut}(CEA)\} > W_{ut}(CEL)$.

Now compare the CEA and the P rules. Assume first that the condition $c_i \geq \frac{E}{n}$ for all i holds. Then, $CEA(c, E) = \left(\frac{E}{n}, ..., \frac{E}{n}\right)$. Therefore, $l(CEA)$ is such that $l_i(CEA)) = c_i - \frac{E}{n}$ for all i. As such, $\sigma^2(l(CEA)) = \sigma^2(c)$. Instead, $P(c, E) = \lambda c$ with $\lambda = \frac{E}{C}$ such that $\lambda \in (0, 1)$. Therefore, $l_i(P) = (1 - \lambda) c_i$ for all i. It follows that $\sigma^2(l(P)) = (1 - \lambda)^2 \sigma^2(c)$ and thus $\sigma^2(l(CEA)) > \sigma^2(l(P))$. If instead $c_i < \frac{E}{n}$ for some i, then $l(CEA)$ is such that $l_i(CEA) = 0$ for some i, whereas $l(P)$ is such that $l_i(P) > 0$ for all i. The relation $\sigma^2(l(CEA)) > \sigma^2(l(P))$ thus still holds. Therefore, $W_{ut}(CEA) > W_{ut}(P)$. Since we already showed that $\min\{W_{ut}(P), W_{ut}(CEA)\} > W_{ut}(CEL)$, we can conclude that $W_{ut}(CEA) > W_{ut}(P) > W_{ut}(CEL)$. ■

Proof of Proposition 2

Let $(c, E) \in \mathbb{B}^N$ and $r = c$. Now consider the problem $\max_l W_{ut} = E + \sum_{i \in N} \mu(-l_i)$ where $l_i = c_i - R_i(c, E) \geq 0$, $l = (l_1, ..., l_n)$, and $\mu(-l_i) \leq 0$ for any $i \in N$. If $C = E$ then $l = (0, ..., 0)$ and the SCF rule (as any other rule) trivially maximizes welfare. If instead $C > E$ then l is such that $l_i > 0$ (and thus $\mu(-l_i) < 0$) for $\xi(l) \in \{1, ..., n\}$ claimants. Let R and R' be two rules and denote with $l(R)$ and $l\left(R'\right)$ the vectors of individual losses. Given that $\sum_{i \in N} l_i(R) = \sum_{i \in N} l_i\left(R'\right) = L$, the diminishing marginal sensitivity to losses of $\mu(\cdot)$ implies that if $\xi(l(R)) < \xi\left(l\left(R'\right)\right)$ then $W_{ut}\left(R\right) > W_{ut}\left(R'\right)$. By construction the SCF rule minimizes $\xi(l(R))$ and thus maximizes utilitarian welfare. ■

Proof of Proposition 3

With no loss of generality, let the problem $(c, E) \in \mathbb{B}^N$ be such that $c_i \leq c_{i+1}$ for any $i \in \{1, ..., n-1\}$. The SCF rule selects the awards vector

$$SCF(c,E) = \left(c_1, c_2, ..., \overbrace{E - \sum_{j=1}^{k-1} c_j}^{SCF_k(c,E) \text{ with } k \in \{1,...,n\}}, 0, ..., 0 \right)$$

where claimant $k \in \{1, ..., n\}$ is such that $\sum_{j=1}^{k-1} c_j < E \leq \sum_{j=1}^{k} c_j$. The awards vector can be analogously expressed as

$$SCF(c,E) = \left(c_1, c_2, ..., c_k - \left(L - \sum_{j=k+1}^{n} l_j \right), 0, ..., 0 \right)$$

where $l_j = c_j - SCF_j(c, E) \geq 0$ is claimant j's loss. The SCF rule thus achieves utilitarian welfare

$$W_{ut}(SCF) = E \quad + \quad \overbrace{\mu \left(- \left(L - \sum_{j=k+1}^{n} l_j \right) \right)}^{\text{perceived loss of } k} \quad + \quad \overbrace{\sum_{j=k+1}^{n} \mu \left(-c_j \right)}^{\text{perceived losses of } j>k}$$

By Proposition 2, this is the maximum level of utilitarian welfare that any rule can achieve. Consider now the awards vector

$$\hat{R}(c,E) = \left(c_1, ..., c_{\hat{k}} - \left(L - \sum_{j=k+1}^{n} l_j \right), c_{\hat{k}+1}, ..., c_k, 0, ..., 0 \right)$$

with $\hat{k} \in \{1, ..., k-1\}$ and $c_{\hat{k}} - \left(L - \sum_{j=k+1}^{n} l_j \right) \geq 0$. Rule $\hat{R}$ achieves utilitarian welfare

$$W_{ut}(\hat{R}) = E \quad + \quad \overbrace{\mu \left(- \left(L - \sum_{j=k+1}^{n} l_j \right) \right)}^{\text{perceived loss of } \hat{k}} \quad + \quad \overbrace{\sum_{j=k+1}^{n} \mu \left(-c_j \right)}^{\text{perceived losses of } j>k}.$$

Therefore, $W_{ut}(\hat{R}) = W_{ut}(SCF)$ and rule $\hat{R}$ also maximizes utilitarian welfare.[24] To compare the two rules in terms of inequality, let $\sigma^2(R) = \frac{\sum_{i=1}^{n} ((R_i(c,E) - E/n))^2}{n}$ be the variance of $R(c, E)$. Then, the condition $\sigma^2(SCF) \leq \sigma^2(\hat{R})$ holds if and only if

[24]The vector $\hat{R}(c, E)$ may not exist, i.e., there might be no agent $\hat{k}$ such that $c_{\hat{k}} - \left(L - \sum_{j=k+1}^{n} l_j \right) \geq 0$. If this is the case, the SCF rule is the unique rule that maximizes utilitarian welfare.

$$\left(c_{\hat{k}} - E/n\right)^2 + \left(c_k - \left(L - \sum_{j=k+1}^{n} l_j\right) - E/n\right)^2 \leq$$

$$\left(c_{\hat{k}} - \left(L - \sum_{j=k+1}^{n} l_j\right) - E/n\right)^2 + \left(c_k - E/n\right)^2$$

since all other terms cancel out. This simplifies to

$$-2\left(L - \sum_{j=k+1}^{n} l_j\right)\left(c_k - c_{\hat{k}}\right) \leq 0$$

which is always true given that $\left(L - \sum_{j=k+1}^{n} l_j\right) \geq 0$ and $c_k \geq c_{\hat{k}}$. ■

Proof of Proposition 4

I first show that if a rule R maximizes utilitarian welfare then the awards vector $R(c, E)$ satisfies *Large Losers* and *Unique Residual Loser*. By the proof of Proposition 3, we know that the rules that maximize utilitarian welfare are those that select an awards vector

$$\hat{R}(c, E) = \left(c_1, ..., c_{\hat{k}} - \left(L - \sum_{j=k+1}^{n} l_j\right), c_{\hat{k}+1}, ..., c_k, 0, ..., 0\right)$$

where claimant $k \in \{1, ..., n\}$ is such that $\sum_{j=1}^{k-1} c_j < E \leq \sum_{j=1}^{k} c_j$, and $\hat{k} \in \{1, ..., k\}$ is such that $c_{\hat{k}} - \left(L - \sum_{j=k+1}^{n} l_j\right) \geq 0$. I now show that $\tilde{L}_i < c_i$ for all $i \in \{1, .., k\}$, whereas $\tilde{L}_i \geq c_i$ for all $i \in \{k+1, .., n\}$, where $\tilde{L}_i$ is claimant i's *Cumulative Aggregate Loss* (see Definition 2 in the main text).

Consider claimant k. Then, $\tilde{L}_k = \sum_{i=1}^{k} c_i - E \geq 0$ because k is the first agent for which the condition $\sum_{i=1}^{k} c_i \geq E$ holds. However, $\tilde{L}_k < c_k$ since $\tilde{L}_k - c_k = \sum_{i=1}^{k-1} c_i - E < 0$. Since the condition $\tilde{L}_i < c_i$ holds for claimant k, it also holds for all $i \in \{1, .., k-1\}$. Consider now claimant $k+1$. Then, $\tilde{L}_{k+1} = \sum_{i=1}^{k+1} c_i - E > 0$ because $c_{k+1} \geq c_k > 0$. However it is now the case that $\tilde{L}_{k+1} \geq c_{k+1}$ since $\tilde{L}_{k+1} - c_{k+1} = \sum_{i=1}^{k} c_i - E = \tilde{L}_k \geq 0$. Since the condition $\tilde{L}_i \geq c_i$ holds for claimant $k+1$, it also holds for all $i \in \{k+2, .., n\}$.

The awards vector $\hat{R}(c, E)$ thus satisfies *Large Losers*, since it assigns $\hat{R}_i(c, E) = 0$ to each claimant $i \in \{k+1, ..., n\}$ and these are the agents for which the condition $\tilde{L}_i \geq c_i$ holds. The vector $\hat{R}(c, E)$ also satisfies *Unique Residual Loser* since claimant $k \in \{1, ..., n\}$ is the agent for which $0 < \tilde{L}_k < c_k$ and claimant $\hat{k} \in \{1, ..., k\}$ is the agent that fulfills the condition $R_{\hat{k}}(c, E) = c_{\hat{k}} - \tilde{L}_k \geq 0$.

I now prove that if an awards vector $R(c, E)$ satisfies *Large Losers* and *Unique Residual Loser* then rule R maximizes utilitarian welfare. Consider the generic awards vector:

$$R(c,E) = (R_1(c,E), ..., R_n(c,E))$$

with $R_i(c,E) \in [0, c_i]$ for any $i \in N$. *Large Losers* implies:

$$R(c,E) = (R_1(c,E), R_2(c,E), ..., R_k(c,E), 0, ..., 0)$$

since claimants $j \in \{k+1, ..., n\}$ are those for which the condition $\tilde{L}_j \geq c_j$ holds. *Unique Residual Loser* then implies that there exists a claimant $\hat{k} \in \{1, ..., k\}$ such that:

$$R(c,E) = \left(R_1(c,E), ..., c_{\hat{k}} - \tilde{L}_k, R_{\hat{k}+1}(c,E), ..., R_k(c,E), 0, ..., 0\right).$$

By construction, $\tilde{L}_k + \sum_{j=k+1}^{n} \tilde{L}_j = L$. *Balance* implies $\sum_{i=1}^{k} R_i(c,E) = E$. *Boundedness* then necessarily leads to:

$$R(c,E) = \left(c_1, ..., c_{\hat{k}} - \left(L - \sum_{j=k+1}^{n} l_j\right), c_{\hat{k}+1}, ..., c_k, 0, ..., 0\right)$$

which is a vector that maximizes utilitarian welfare (see the proof of Proposition 3). ∎

Proof of Proposition 5

It is immediate to verify that the *SCF* rule satisfies *Large Losers* and *Unique Residual Loser Is The Last*. I now prove that the converse also holds true. As in the proof of Proposition 4, *Large Losers* implies:

$$R(c,E) = (R_1(c,E), R_2(c,E), ..., R_k(c,E), 0, ..., 0)$$

where $k \in \{1, ..., n\}$ is such that $0 < \tilde{L}_k < c_k$. *Unique Residual Loser Is The Last* then implies:

$$R(c,E) = \left(R_1(c,E), R_2(c,E), ..., c_k - \tilde{L}_k, 0, ..., 0\right).$$

Since $\tilde{L}_k + \sum_{j=k+1}^{n} \tilde{L}_j = L$, *Boundedness* and *Balance* then necessarily require:

$$R(c,E) = \left(c_1, c_2 ..., c_k - \left(L - \sum_{j=k+1}^{n} l_j\right), 0, ..., 0\right)$$

and given that $c_k - \left(L - \sum_{j=k+1}^{n} l_j\right) = E - \sum_{j=1}^{k-1} c_j$, the awards vector can be rewritten as

$$R(c,E) = \left(c_1, c_2, ..., E - \sum_{j=1}^{k-1} c_j, 0, ..., 0\right)$$

which is the *SCF* solution. ■

Proof of Proposition 6

The utility function of any claimant $i \in N$ is continuous and strictly increasing in $R_i(c, E)$ and the awards vector satisfies *Balance*. Then, if feasible, maxmin welfare gets maximized by the vector $R(c, E)$ such that $\min\{u(R_i(c, E) \mid r_i = c_i)\}_{i \in N} = \max\{u(R_i(c, E) \mid r_i = c_i)\}_{i \in N}$. If instead *Boundedness* makes such a vector unfeasible then maxmin welfare gets maximized by any vector $R(c, E)$ with $R_j(c, E) = c_j$ where agent $j \in N$ is the agent for which $u(c_j \mid r_j = c_j) = c_j = \min\{u(R_i(c, E) \mid r_i = c_i)\}_{i \in N}$. By construction, the *MUG* rule selects these award vectors in both cases and thus it always maximizes welfare. ■

Proof of Proposition 7

Let $(c, E) \in \mathbb{B}^N$ and $r = \mathbf{0}$. Generic rule R generates utilitarian welfare $W_{ut}(R) = E + \sum_{i \in N} \mu(g_i(R))$, where $g_i(R) = R_i(c, E) - 0 = R_i(c, E)$ is claimant i's perceived gain. It follows that $g_i(R) \geq 0$ for all $i \in N$ and $\sum_{i \in N} g_i(R) = E$.
Because of the strict concavity of $\mu(\cdot)$ in the domain of gains, the lower is the variance of $g(R) = (R_1(c, E), ..., R_n(c, E))$, the higher is the welfare that R generates. Since the *CEA* rule allocates E as equally as possible, the awards vector $CEA(c, E)$ maximizes welfare and thus $W_{ut}(CEA) > \max\{W_{ut}(P), W_{ut}(CEL)\}$ whenever $c_i \neq c_j$ for some $i, j \in N$.
Now compare the P and the CEL rules. Assume first that $c_i \geq \frac{L}{n}$ for all $i \in N$. Then, $g_i(P) = \lambda c_i$ (with $\lambda \in (0, 1)$) and $g_i(CEL) = c_i - \frac{L}{n}$ for all i. Thus, $\sigma^2(g(P)) < \sigma^2(g(CEL))$ since $\sigma^2(g(P)) = \lambda^2 \sigma^2(c)$ whereas $\sigma^2(g(CEL)) = \sigma^2(c)$. If instead $c_i < \frac{L}{n}$ for some $i \in N$, then $g_i(CEL) = 0$ for some $i \in N$ whereas $g_i(P) > 0$ for all $i \in N$ such that the relation $\sigma^2(g(P)) < \sigma^2(g(CEL))$ still holds. Therefore, $W_{ut}(CEA) > W_{ut}(P) > W_{ut}(CEL)$. ■

Proof of Proposition 8

Let $(c, E) \in \mathbb{B}^N$ and $r = m$. Then $W_{ut}(R) = E + \sum_i \mu(g_i(R))$ where $g_i(R) = R_i(c, E) - m_i \geq 0$ for all $i \in N$. The strict concavity of $\mu(\cdot)$ in the domain of gains implies that the lower is the variance of $g(R)$, the higher is $W_{ut}(R)$.
Given the order $\prec_c$, the *CEG* rule implements the awards vector

$$CEG(c, E) = \left(c_1, ..., c_{j-1}, m_j + \xi, ..., m_n + \xi\right)$$

where $\xi = \frac{E - \sum_{i \succeq_c j} m_i - \sum_{i \prec_c j} CEG_i}{n - j + 1}$ and claimant $j \in \{1, ..., n\}$ is the first agent for which the condition $m_j + \xi \leq c_j$ holds. Therefore, the vector of perceived gains is

$$g(CEG) = \left(c_1 - m_1, ..., c_{j-1} - m_{j-1}, \xi, ...\xi\right).$$

such that $g_i(CEG) < \xi$ for all $i \in \{1, ..., j-1\}$ and the last $n - j + 1$ terms are equal. *Boundedness* implies that there are no awards vector $R(c, E)$ such that $g_i(R) > g_i(CEG)$

for some $i \in \{1, ..., j-1\}$. *Balance* implies that if an awards vector $R(c, E)$ is such that $g_i(R) < \xi$ for some $i \in \{j, ..., n\}$ then it must be the case that $g_k(R) > \xi$ for some $k \in \{j, ..., n\}$ with $k \neq i$. Therefore, $\sigma^2(g(CEG)) < \sigma^2(g(R))$ for any $R \neq CEG$ so that the CEG rule maximizes utilitarian welfare. ■

Proof of Proposition 9

The proof replicates the proof of Proposition 6 with the condition $r_i = m_i$ instead of $r_i = c_i$. ■

Proof of Proposition 10

Let $(c, E) \in \mathbb{B}^N$. Let $r = F$ and F be such that $f(R(c, E)) = 1$. Then, $W_{ut}(R) = E$ since $r_i = R_i(c, E)$ and thus $\mu(R_i(c, E) - r_i) = \mu(0) = 0$ for all $i \in N$. Now consider any rule $R' \neq R$. Since both $R(c, E)$ and $R'(c, E)$ satisfy *Balance*, it must be the case that $R'_i(c, E) > R_i(c, E)$ for some $i \in N$ and $R'_j(c, E) < R_j(c, E)$ for some $j \in N$ with $j \neq i$. However, the fact that $|\mu(-z)| > \mu(z)$ for any $z > 0$ (losses loom larger than gains) implies that $\sum_{i \in N} \mu\left(R'_i(c, E) - R_i(c, E)\right) < 0$ and thus $W_{ut}(R') < E$. Therefore, $W_{ut}(R) > W_{ut}(R')$ for any $R' \neq R$. ■

Proof of Proposition 11

Let $(c, E) \in \mathbb{B}^N$. Let $r = F$ and F be such that $f(CEA(c, E)) = 1$. Define agent 1 as an agent for which $c_1 \leq c_i$ for any $i \in N$. Then, $W_{mm}(CEA) = CEA_1(c, E)$ since, for any $i \in N$, $u(CEA_i(C, E) \mid (CEA_i(c, E)) = CEA_i(C, E)$ and $CEA_1(c, E) \leq CEA_i(c, E)$ because the CEA rule satisfies the property of *Order Preservation in Gains*. Any rule $R \neq CEA$ leads instead to $W_{mm}(R) = R_1(c, E) + \mu(R_1(c, E) - CEA_1(c, E))$. Given that $R_1(c, E) \leq CEA_1(c, E)$ and $\mu(R_1(c, E) - (CEA(c, E)) \leq 0$, it follows that $W_{mm}(CEA) \geq W_{mm}(R)$ for any $R \neq CEA$. ■

References

[1] Abeler, J., Falk, A., Goette, L. and Huffman, D., 2011. Reference points and effort provision, *American Economic Review*, 101, 470-492.

[2] Aumann, R.J., Maschler, M., 1985. Game theoretic analysis of a bankruptcy problem from the Talmud, *Journal of Economic Theory*, 36, 195-213.

[3] Bernheim, B.D., Rangel, A., 2007. Behavioral public economics: Welfare and policy analysis with non-standard decision makers. In: Diamond, P., Vartiainen, H. (Eds.): *Behavioral economics and its applications*. Princeton, NY: Princeton University Press.

[4] Bernheim, B.D., Rangel, A., 2009. Beyond revealed preference: Choice-theoretic foundations for behavioral welfare economics, *The Quarterly Journal of Economics*, 124, 51-104.

[5] Bolton, G.E., Brandts, J., Ockenfels, A., 2005. Fair procedures: Evidence from games involving lotteries, *Economic Journal*, 115, 1054-1076.

[6] Bosmans, K., Lauwers, L., 2011. Lorenz comparisons of nine rules for the adjudication of conflicting claims, *International Journal of Game Theory*, 40, 791-807.

[7] Chambers, C.P., Moreno-Ternero, J., 2017. Taxation and poverty, *Social Choice and Welfare*, 48, 153-175.

[8] Chun, Y., Thomson, W., 1992. Bargaining problems with claims, *Mathematical Social Sciences*, 24, 19-33.

[9] Ericson, K., Fuster, A., 2011. Expectations as endowments: Evidence on reference-dependent preferences from exchange and valuation experiments, *Quarterly Journal of Economics*, 126, 1879-1907.

[10] Fleurbaey, M., Schokkaert, E., 2013. Behavioral welfare economics and redistribution, *American Economic Journal: Microeconomics*, 5, 180-205.

[11] Gravel, N., Moyes, P., 2013. Utilitarianism or welfarism: does it make a difference?, *Social Choice and Welfare*, 40, 529-551.

[12] Gruber, J., Kőszegi, B., 2004. Tax incidence when individuals are time inconsistent: The case of cigarette excise taxes, *Journal of Public Economics*, 88, 1959-1987.

[13] Gupta, S., Livne, Z., 1988. Resolving a conflict situation with a reference outcome: An axiomatic model, *Management Science*, 34, 1303-1314.

[14] Habis, H., Herings, P.J.J., 2013. Stochastic bankruptcy games, *International Journal of Game Theory*, 42, 973-988.

[15] Herrero, C., 1998. Endogenous reference points and the adjusted proportional solution for bargaining problems with claims, *Social Choice and Welfare*, 15, 113-119.

[16] Herrero, C., Villar, A., 2001. The three musketeers: Four classical solutions to bankruptcy problems, *Mathematical Social Sciences*, 39, 307-328.

[17] Herrero, C., Villar, A., 2010. The rights egalitarian solution for NTU problems, *International Journal of Game Theory*, 39, 137-150.

[18] Hougaard, J.L., Moreno-Ternero, J., Østerdal, L.P., 2012. A unifying framework for the problem of adjudicating conflicting claims, *Journal of Mathematical Economics*, 48, 107-114.

[19] Hougaard, J.L., Moreno-Ternero, J., Østerdal, L.P., 2013a. Rationing in the presence of baselines, *Social Choice and Welfare*, 40, 1047-1066.

[20] Hougaard, J.L., Moreno-Ternero, J., Østerdal, L.P., 2013b. Rationing with baselines: The composition extension operator, *Annals of Operations Research*, 211, 179-191.

[21] Kahneman, D., Tversky, A., 1979. Prospect theory: An analysis of decision under risk, *Econometrica* 47, 263-291.

[22] Kőszegi, B., Rabin, M., 2006. A model of reference-dependent preferences, *Quarterly Journal of Economics* 121, 1133-1165.

[23] Kőszegi, B., Rabin, M., 2007. Reference-dependent risk attitudes, *American Economic Review* 97, 1047-1073.

[24] Mariotti, M., Villar, A., 2005. The Nash rationing problem, *International Journal of Game Theory*, 33, 367-377.

[25] Moulin, H., 1987. Equal or proportional division of a surplus, and other methods, *International Journal of Game Theory*, 16, 161-186.

[26] Moulin, H., 2000. Priority rules and other asymmetric rationing methods, *Econometrica*, 68, 643-684.

[27] Moulin, H., 2002. Axiomatic cost and surplus-sharing. In: Arrow, K., Sen, A., Suzumura, K. (Eds.): *The Handbook of Social Choice and Welfare*, Vol. 1. Amsterdam: Elsevier.

[28] Moulin, H., 2003. *Fair Division and Collective Welfare*, MIT Press: Cambridge, MA, USA.

[29] O'Donoghue, T., Rabin, M., 2006. Optimal sin taxes, *Journal of Public Economics*, 90, 1825-1849.

[30] O'Neill, B., 1992. A problem of rights arbitration from the Talmud, *Mathematical Social Sciences*, 2, 345-371.

[31] Pulido, M., Borm, P., Hendrichx, R., Llorca, N., Sanchez-Soriano, J., 2008. Compromise solutions for bankruptcy situations with references, *Annals of Operations Research*, 158, 133-141.

[32] Pulido, M., Sanchez-Soriano, J., Llorca, N., 2002. Game theory techniques for university management: an extended bankruptcy model, *Annals of Operations Research*, 109, 129-142.

[33] Shalev, J., 2000. Loss aversion equilibrium, *International Journal of Game Theory*, 29, 269-287.

[34] Thomson, W., 1994. Cooperative models of bargaining. In: Aumann, R.J., Hart, S. (Eds.): *The Handbook of Game Theory*, Vol. 2. Amsterdam: Elsevier.

[35] Thomson, W., 2003. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: A survey, *Mathematical Social Sciences*, 45, 249-297.

[36] Thomson, W., 2015. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: An update, *Mathematical Social Sciences*, 74, 41-59.

[37] Thomson, W., Yeh, C.-H., 2008. Operators for the adjudication of conflicting claims, *Journal of Economic Theory*, 143, 177-198.

[38] Young, H.P., 1988. Distributive justice in taxation, *Journal of Economic Theory*, 48, 321-335.